BIEN 314

Coursepack

Transport Phenomena in Biological Systems 1

Prepared by

BUSS

Professor Caroline Elizabeth Wagner

Preface

Welcome to the BIEN 314 — Transport Phenomena in Biological Systems 1 official course-pack! This document has been prepared by the Bioengineering Undergraduate Student Society (BUSS) to best assist you as you make your way through the course. The course-pack addresses the material according to an objective-based approach. This means that key concepts from each lecture are listed and expanded upon.

The course-pack is organized by topic taught in the course. There is course material information, such as explanations, notes, custom figures, and examples organized to directly respond to each lecture's topics.

Please note that this document was created using source material from the Fall 2023 semester. Due to the ever-changing nature of courses offered by the Department of Bioengineering at McGill, it is possible that parts of this course-pack no longer accurately reflect the current contents or organization of the course. In case of doubt, refer directly to the course instructor, Professor Caroline Elizabeth Wagner.

All figures, unless otherwise indicated as from Professor Wagner's slides, are adapted from the textbook *Biotransport: Principles and Applications* by Roselli & Diller, 2011.

This course-pack was written and designed by Syphax Ramdani, with the help of BUSS Vice-President of Academics Anna Shi (2024 — 2025). This course-pack could not have been produced without the help of Professor Caroline Elizabeth Wagner.

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Contents

Introduction and General Concepts

What are Transport Phenomena?

Transport phenomena, put simply, is the study of how certain things move. In BIEN 314, these things are momentum, seen in the fluid dynamics portion, and energy or heat, seen in the heat transfer portion. However, the field extends beyond these. For example, mass transfer will be taught in BIEN 340 and deals with the transport of “mass”, typically as dilute solutes. In fact, you may have already seen transport phenomena of electrical charges in circuits classes.

These things are grouped into transport phenomena because they behave similarly. Take the example of heat and mass transfer. Let’s say I had a cup of tea with a sugar cube at the bottom, and, as a perfectly normal and sane bioengineering student, I heated up my cup of tea on a hot plate. A kind of “diffusion” happens for both sugar and heat as they move from the bottom to the top of the cup through the random motion of water and sugar molecules. Mathematically, the equations are also similar: Fick’s law of diffusion and Fourier’s law of conduction are nearly identical. There are many other similarities that you will see throughout the course, though of course each topic has its own quirks.

Fick’s law of diffusion	**Fourier’s law of conduction**
$\boldsymbol{J = -D\frac{dc}{dy}}$	$q = -k\frac{dT}{dx}$
J: molar flux of species in y direction	*q*: heat flux in *x* direction
c: concentration of species	*T*: temperature
D: diffusion coefficient of species	*k*: thermal conductivity

Basic Concepts and Definitions

This is a summary of basic concepts which are useful for this class. You’ve already seen most of them in thermodynamics.

- **System**: portion of the universe we are interested in.
- **Environment**: rest of the universe.
- **Boundary**: interface between system and environment.
  - *Isolated* system: no exchange of energy or matter between system and environment
  - *Closed* system: exchange of energy, but not matter between system and environment

- *Open* system: exchange of energy and matter between system and environment
- **Property**: a characteristic of a system.
    - *Intensive* properties: independent of system size (temperature, pressure, density, etc.)
    - *Extensive* properties: dependent on system size and can enter or leave the system (mass, heat, volume, etc.)
- **Continuum**: A body can be considered a continuum if you can treat it as being made of a continuous substance rather than as a collection of discrete particles. We want to treat bodies as continuums because dealing with atoms is hard.
- **Equilibrium**: no net exchange of mass, energy, or any other extensive property. Homogeneous temperature and pressure.

Diffusive Transport

One of the main mechanisms for transport is caused by the **random motion of molecules**. We'll refer to it as "diffusion", even though it has many names depending on the specific topic: for momentum transport, it's related to viscosity; for heat transfer, it's called heat conduction; and for mass transfer, it's called diffusion.

In all these cases, the quantity moves from a region where it is more "concentrated" to a region where it is less "concentrated" due to random thermal motion. Let's take the example of the cup of tea again, where the bottom of my cup is hot and the top is cold. Due to random motion, the hot water molecules will move up and down the cup. Though some molecules might return to the bottom, others will keep moving up, carrying heat upwards with them. After some time has passed, the hot and cold water molecules will be in completely random positions inside my cup, so the bottom and top will have the same amount of hot and cold water molecules, such that the temperature of my cup is now uniform. This logic applies to momentum and mass as well.

Mathematically, the equations for these diffusive processes look similar as well. To start, we will briefly define flow rate and flux. The flow rate of a quantity X is the amount of X that passes through an area A per unit time. The flux of X is the flow rate per unit area. It is typically a vector quantity, so flow rate and flux are related by a surface integral. Let's take volume as an example. The volumetric flow rate (Q_v) through a surface S of area A is related to the volumetric flux (or simply velocity, $\vec{v}$) through the equation

$$Q_v = \iint_S \vec{v} \cdot \hat{n} \, dA$$

where $\hat{n}$ is the unit vector normal to the surface. If the flux is normal to the surface, then the equation is simply

$$Q_v = \langle v \rangle A$$

where $\langle v \rangle$ is the average velocity.

In diffusive transport, flux is typically proportional to the negative gradient of some intensive property, called potential. A constitutive relationship relates the flux of an extensive property to the gradient of an intensive property. These constitutive relationships typically have the following form:

$$Diffusive\ flux\ in\ the\ x\ direction = (constitutive\ property)\left(-\frac{\delta(potential)}{\delta x}\right)$$

Taking the example of heat transfer, we know that heat goes from regions with high temperature to regions with low temperature, so our potential is temperature. We also know that heat travels faster in materials with high thermal conductivity, which will be our constitutive property. Putting it all together, we come back to Fourier's law of conduction, seen in the "Heat" column in the table below.

Table 1: Diffusive transport.

	MOMENTUM	**HEAT**	**MASS (BIEN 340)**
FLUX	x-momentum flux in the y-direction τ_{yx}	Heat flux q	Molar flux J
POTENTIAL	Velocity in the x-direction v_x	Temperature T	Molar concentration c
CONSTITUTIVE PROPERTY	Viscosity μ	Thermal conductivity k	$Diffusivity$ D
CONSTITUTIVE RELATION	$\tau_{yx} = \mu \frac{\delta v_x}{\delta y}$	$q = k \frac{\delta T}{\delta x}$	$J = D \frac{\delta c}{\delta x}$

Convective Transport

The other main method for transport is convective transport. Rather than being driven by random motion, this mechanism is driven by the flow of fluid containing the quantity, or **bulk fluid motion**. If I were to take my hot cup of tea and pour it into another cold cup of tea, the heat in the first cup would be transferred through the bulk motion of tea to the second cup.

Mathematically, since these are driven by fluid flow, the flux is generally the quantity of interest per unit volume multiplied by the fluid velocity.

$$Convective\ flux\ in\ x\ direction = (quantity\ per\ volume)(fluid\ velocity\ in\ x\ direction)$$

Again, across different quantities, the equations look similar.

Table 2: Convective transport.

	MOMENTUM	**HEAT**	**MASS (BIEN 340)**
QUANTITY PER VOLUME	Specific momentum pv_x	Specific thermal energy $\rho c_p(T - T_R)$	Concentration C
FLUX	ρv_x^2	$\rho c_p(T - T_R)v_x$	Cv_x

Dimensional Analysis

Buckingham Pi Theorem

Fundamental dimensions

A fundamental dimension can be thought of as a type of measurement. For example, **mass**, **length** and **time** can be fundamental dimensions. On the other hand, velocity can be obtained from other fundamental dimensions, specifically as distance/time, so it wouldn't be a fundamental dimension. Here is a table of fundamental dimensions (and their units) for the International System Of Units (SI) system.

Table 3: Fundamental dimensions and SI units.

Dimension	Symbol	SI unit
Mass	M	kilogram (kg)
Length	L	metre (m)
Time	T	second (s)
Amount of substance	N	mole (mol)
Temperature	Θ	Kelvin (K)
Electric current	I	ampere (A)
Luminous intensity (you will probably never use this)	J	Candela (cd)

Technically, we could choose different fundamental dimensions. For example, we could decide that velocity and distance are fundamental dimensions, but time is not, because it can be obtained as distance/velocity. However, this would just cause confusion. The SI system makes the most sense and is the easiest to use, so just use that one.

Definition and use

The Buckingham Pi theorem states that a function f of v variables $x_1, x_2, x_3, \dots, x_v$:

$$x_1 = f(x_2, \dots, x_v)$$

can be rewritten in terms of p dimensionless variables $\pi_1, \pi_2, \dots, \pi_p$ called **pi groups**, which are formed from the original variables:

$$\pi_1 = F\left(\pi_2, \dots, \pi_p\right)$$

Note that these pi groups are NOT UNIQUE in general, so multiple sets of pi groups are possible. The number of pi groups is $p = v - d$, where v is the number of original variables and d is the number of fundamental dimensions the problem has.

In short, using the theorem, we can **reduce the number of variables of an equation** by the number of fundamental dimensions present. The important part is that you do not need to know the function itself to use this theorem. This is useful for experiments, since it allows you to reduce the number of variables to test. Very often, problems in transport phenomena involve many variables and do not have simple equations. Using this theorem, empirical relations can be found. A good example of where Buckingham Pi is used is in finding the heat transfer coefficient in the convective heat transfer portion.

General procedure

1. Identify the v independent variables $x_1, x_2, \ldots, x_v$ in the problem.
2. List out the fundamental dimensions for each independent variable and count how many fundamental dimensions d there are. You will have to make $p = v - d$ pi groups. It's useful to make a table listing the dimension exponents for each variable.
3. Separate the variables into p dependent (excluded) variables and d independent (core) variables.
    - Your variables of interest should be in the dependent variables.
    - Any already dimensionless variables should be in the dependent variables (and will form pi groups on their own).
    - Your independent variables should:
        i. Contain all fundamental dimensions present in the problem.
        ii. No two should have the same fundamental dimensions nor multiples of the same fundamental dimensions (ex: can't have length (L) and area (L^2)).
        iii. Typically, but not always, there is one variable for a fluid property, one for flow geometry, and one for flow rate.
4. Construct p dimensionless groups, each from one of the dependent variables and all of the independent variables. A pi group should be of the form $\pi_i = x_i y_1^a y_2^b y_3^c \ldots$, where x_i is one of the dependent variables, $y_{1,2,3,\ldots}$ are the independent variables, and $a, b, c, \ldots$ are exponents which have not been determined yet.
5. Find the exponent values of the independent variables to make the group dimensionless.

Simple example

An object of mass m is attached to a spring of stiffness k at a distance l from its equilibrium position. If we release the object, we wish to know the period t for the object's oscillation. The relation between variables can be expressed as:

$$t = f(m, k, l)$$

1. There are $v = 4$ variables: mass m, stiffness k, distance l, and period t (variable of interest).
2. The dimensions are: $\dim(m) = M$, $\dim(k) = MT^{-2}$, $\dim(l) = L$, $\dim(t) = T$

It sometimes helps to make a table for this step, which lists the exponents for all fundamental dimensions of each variable:

Variable	Time (T)	Length (L)	Mass (M)
mass m	0	0	1
stiffness k	-2	0	1
distance l	0	1	0
period t	1	0	0

There are $d = 3$ fundamental dimensions: time, length, and mass, meaning there is only $p = 4 - 3 = 1$ pi group.

3. The one dependent variable must be the period, as it is the variable of interest. The other three variables are the independent variables. They encompass all fundamental dimensions, and no two variables have the same (or multiples of the same) fundamental dimensions, so we can proceed.
4. The pi group will be $\pi_1 = tm^a k^b l^c$
5. To find the exponents the group must be dimensionless. Thus, we must find the dimension of the pi group in terms of the variables a, b, c:

$$Dim(\pi_1) = (T)(M)^a(MT^{-2})^b(L)^c = T^{1-2b}M^{a+b}L^c$$

The group must be dimensionless:

$$Dim(\pi_1) = T^{1-2b}M^{a+b}L^c = T^0M^0L^0$$

Thus, we solve the system of equations:

$$\begin{cases} 1 - 2b = 0 \\ a + b = 0 \\ c = 0 \end{cases}$$

and we find that $a = -\frac{1}{2}, b = \frac{1}{2}, c = 0$. Our pi group is $\pi_1 = t\sqrt{\frac{k}{m}}$.

We can use this to rewrite our original function to $F\left(t\sqrt{\frac{k}{m}}\right) = 0$, so $t\sqrt{\frac{k}{m}} = C$, where C is some constant. Thus, we know that $t = C\sqrt{\frac{m}{k}}$. The real equation for the period of oscillation of a mass on a spring is $t = 2\pi\sqrt{\frac{m}{k}}$, so our analysis makes sense.

Another example

Determine the specific energy (energy per unit mass) lost due to friction Ê when a fluid of density ρ and viscosity μ goes through a rectangular conduit of length l, width w, and aspect ratio α with average velocity v.

1. There are $v = 7$ variables: density ρ, viscosity μ, length l, width w, aspect ratio α, average velocity v, and energy loss Ê (variable of interest).
2. The dimensions are:

Variable	Time (T)	Length (L)	Mass (M)
Density ρ	0	-3	1
Viscosity μ	-1	-1	1
Length l	0	1	0
Width w	0	1	0
Aspect ratio α	0	0	0
Average velocity v	-1	1	0
Specific energy $\hat{E}$	-2	2	0

There are $d = 3$ fundamental dimensions: time, length, and mass, meaning there are $p = 7 - 3 = 4$ pi groups.

3. Finding the dependent and independent variables:
   - Dependent variables:
     i. Specific energy lost due to friction is the variable of interest
     ii. Aspect ratio is already dimensionless
     iii. Since length and width have the same dimensions, one will be independent and the other dependent. Let's choose length as the dependent variable.

iv. As the final variable, since both density and viscosity characterize fluid properties, let's split them between dependent and independent variables. For the dependent variables, let's choose density.

- Independent variables:
  i. For fluid properties, let's use viscosity.
  ii. For flow geometry, let's choose width
  iii. For flow rate, let's use average velocity.

4. $\pi_{\hat{E}} = \hat{E}\mu^a w^b v^c, \pi_\alpha = \alpha\mu^a w^b v^c, \pi_l = l\mu^a w^b v^c, \pi_\rho = \rho\mu^a w^b v^c$
5. We go through the pi groups one by one:

$$Dim(\pi_{\hat{E}}) = (L^2T^{-2})(ML^{-1}T^{-1})^a(L)^b(LT^{-1})^c = T^{-2-a-c}M^aL^{2-a+b+c} = 1$$

$$\begin{cases} -2-a-c=0 \\ a=0 \\ 2-a+b+c=0 \end{cases}$$

$$a=0, b=0, c=-2$$

$$\pi_{\hat{E}} = \hat{E}v^{-2}$$

$$Dim(\pi_\alpha) = (ML^{-1}T^{-1})^a(L)^b(LT^{-1})^c = T^{-a-c}M^aL^{-a+b+c} = 1$$

$$\begin{cases} -a-c=0 \\ a=0 \\ -a+b+c=0 \end{cases}$$

$$a=0, b=0, c=0$$

$$\pi_\alpha = \alpha$$

$$Dim(\pi_l) = (L)(ML^{-1}T^{-1})^a(L)^b(LT^{-1})^c = T^{-a-c}M^aL^{1-a+b+c} = 1$$

$$\begin{cases} -a-c=0 \\ a=0 \\ 1-a+b+c=0 \end{cases}$$

$$a=0, b=-1, c=0$$

$$\pi_l = lw^{-1}$$

$$Dim(\pi_\rho) = (ML^{-3})(ML^{-1}T^{-1})^a(L)^b(LT^{-1})^c = T^{-a-c}M^{1+a}L^{-3-a+b+c} = 1$$

$$\begin{cases} -a-c=0 \\ 1+a=0 \\ -3-a+b+c=0 \end{cases}$$

$$a=-1, b=1, c=1$$

$$\pi_\rho = \rho\mu^{-1}w^1v^1 = \frac{\rho w v}{\mu} = Reynolds\ number$$

This means that:

$$\frac{\hat{\mathrm{E}}}{v^2} = f\left(\alpha, \frac{l}{w}, \frac{\rho w v}{\mu}\right)$$

Now, we could perform experiments varying the aspect ratio, length/width ratio, and Reynolds number to find an equation for $\frac{\hat{\mathrm{E}}}{v^2}$.

Scaling

Explanation and use

The Buckingham pi theorem is useful when the underlying equation is unknown. However, a different problem can happen: the equation is known but is an unsolvable monster. Take the Navier-Stokes equations: while they can describe the motion of incompressible fluids in a wide range of cases, there is a million-dollar standing bounty to anyone who can find whether a general solution even *exists*. Of course, in this class, you will not have to solve million-dollar problems. But you will have to **reduce complicated equations to their simplest form**. This is where scaling comes in.

You will often be able to make assumptions about the problem based on the scale of things. For example, the length of a pipe might be much longer than its width, or the Reynolds number might be small. Using scaling, we create dimensionless parameters from the variables in the equation, to ensure they are of order 10^0. We can then use our assumptions to eliminate terms from the problem. It's easier to understand with an example.

Example

Let's take a 1-dimensional steady-state heat transfer example. A hot liquid at temperature T_1 enters a conduit of length l with velocity v_x and exits the other end at temperature T_2. The liquid has density ρ, viscosity μ, thermal conductivity k, and specific heat capacity c_p. The walls of the conduit are insulated.

Here, heat transfer occurs through two mechanisms. The first is convection, or the bulk movement of hot liquid from one end of the conduit to the other. The second is heat conduction or diffusion: heat moving through the liquid through random molecular movement. We want to know under which conditions one mechanism can be ignored compared to the other.

Logically, we could say that when the liquid is slow, diffusion would be more important. After all, if the liquid were immobile, only diffusion would happen. Similarly, if it were fast, convection would be more important. But we need to know fast compared to *what*. If the velocity were 1 m/s, we would be unable to say whether that is fast or slow. This is where scaling comes in.

The equation for this situation, which you will discover towards the end of the course, is:

$$0 = -\rho c_p v_x \frac{\delta T}{\delta x} + k \frac{\delta^2 T}{\delta x^2}$$

where T is the temperature.

The equation might look a little arcane for now, but what is important to understand is that the $\rho c_p v_x \frac{\delta T}{\delta x}$ term represents convection and the $k \frac{\delta^2 T}{\delta x^2}$ term represents conduction.

To scale the problem, we must create nondimensional numbers for the variables v_x, T, and x. To do so, we need to divide them by some constant in the problem such that they roughly are of order 10^0. If the nondimensional numbers are of order 10^0, we can ignore them when comparing the scale of other terms.

For v_x, we can take the average velocity of the liquid $\langle v \rangle$ as our denominator to create the variable:

$$v^* = \frac{v_x}{\langle v \rangle}$$

For T, we can create

$$T^* = \frac{T - T_1}{T_2 - T_1}$$

which varies from 0 to 1 between the entrance and the exit.

Finally, for x, we can simply compare it to the length of the conduit:

$$x^* = \frac{x}{l}$$

Now, we can substitute them into the starting equation. To make this process clearer, here are the substitutions for each variable.

$$v_x = \langle v \rangle v^*$$

$$\delta T = (T_2 - T_1)\delta T^*, \delta^2 T = (T_2 - T_1)\delta^2 T^*$$

$$\delta x = l\delta x^*, \delta x^2 = l^2 \delta x^{*2}$$

Now to substitute these into the starting equation:

$$0 = -\rho c_p \langle v \rangle v^* \frac{(T_2 - T_1)}{l\delta} \frac{\delta T^*}{x^*} + k \frac{(T_2 - T_1)}{l^2} \frac{\delta^2 T^*}{\delta x^{*2}}$$

$$0 = -\frac{\rho c_p (T_2 - T_1)\langle v \rangle}{l} v^* \frac{\delta T^*}{\delta x^*} + \frac{k(T_2 - T_1)}{l^2} \frac{\delta^2 T^*}{\delta x^{*2}}$$

Rearranging:

$$\frac{\rho c_p l \langle v \rangle}{k} v^* \frac{\delta T^*}{\delta x^*} = \frac{\delta^2 T^*}{\delta x^{*2}}$$

Let's take a closer look at this $\frac{\rho c_p l \langle v \rangle}{k}$ term. First, this term is the Péclet number, which is a BIEN 340 concept, but it compares convective and conductive transport, which is what we are trying to do in this problem.

If the Péclet number is very small or approximately 0, the entire left-hand side of the equation, the convection term, is approximately 0 and can be ignored. This is because the entire $v^* \frac{\delta T^*}{\delta x^*}$ is of order 10^0 regardless, so multiplying something very small by ~ 1 will still result in something very small, i.e. approximately 0. Then, in this case diffusion is much more important. The equation becomes:

$$0 = k \frac{\delta^2 T}{\delta x^2}$$

which is much easier to solve.

On the other hand, if the Péclet number is very large, we move it to the other side.

$$v^* \frac{\delta T^*}{\delta x^*} = \left(\frac{1}{\frac{\rho c_p l \langle v \rangle}{k}} \right) \frac{\delta^2 T^*}{\delta x^{*2}}$$

Since the Péclet number is very large, its inverse will be very small, so the right-hand side of the equation, the diffusion term, can be ignored and convection is much more important. The equation then becomes:

$$0 = -\rho c_p v_x \frac{\delta T}{\delta x}$$

which is, again, much easier to solve.

Some tips:

- Dealing with **derivatives**: let's take $a^* = \frac{a}{A}$ and $b^* = \frac{b}{B}$. To scale the derivative $\frac{\delta^n a}{\delta b^n}$, the equation is:

  $$\frac{\delta^n a}{\delta b^n} = \frac{A}{B^n} \frac{\delta^n a^*}{\delta {b^*}^n}$$

  Essentially, you can find the differentials and treat the derivatives as fractions when you do substitution. You might make a mathematician angry, but it works.
- If you are dealing with **multiple equations**, you might have to find that one variable scales with another and replace it in some other equation. For example, you might end up with an equation like $\frac{\langle v_a \rangle}{A} \frac{\delta v_a^*}{\delta a^*} = \frac{\langle v_b \rangle}{B} \frac{\delta v_b^*}{\delta b^*}$. In this case, you can say that $\langle v_a \rangle \sim \frac{A}{B} \langle v_b \rangle$, and you can replace $\langle v_a \rangle$ with $\frac{A}{B} \langle v_b \rangle$ in other equations.

Scaling Navier-Stokes

Let's scale the Navier-Stokes equation. First, the Navier-Stokes equation can be written in different ways. The professor likely wrote the Navier-Stokes equation in the form:

$$\rho \left(\frac{\delta \vec{v}}{\delta t} + \vec{v} \cdot \nabla \vec{v} \right) = \mu \nabla^2 \vec{v} - \nabla P + \rho \vec{g}$$

where ρ is density, $\vec{v}$ is the fluid velocity vector field, μ is viscosity, P is pressure, and $\vec{g}$ is the acceleration due to gravity.

Here, the problem will be simplified a little. Only the x direction will be considered. This makes the Navier stokes equation look like:

$$\rho \frac{\delta v_x}{\delta t} + \rho v_x \frac{\delta v_x}{\delta x} = \mu \frac{\delta^2 v_x}{\delta x^2} - \frac{\delta P}{\delta x} + \rho g$$

where v_x is x-direction velocity.

In practice, the only difference is that ∇ was replaced by $\frac{\delta}{\delta x}$ and $\vec{v}$ was replaced by v_x. The process of scaling should be the same in either equation. The simplification is only to make it easier to understand and use more familiar notation.

Next, we create our nondimensional parameters. We must create one for velocity v_x, time t, location x, and pressure P.

For velocity, we can scale it with the average velocity $\langle v \rangle$:

$$v^* = \frac{v_x}{\langle v \rangle}$$

The position can be scaled according to some characteristic length L, which could be the length of the problem. All that matters is that it is of the same order of magnitude as the position x.

$$x^* = \frac{x}{L}$$

For time, we can consider the time it would take for a particle of fluid to cross the characteristic length, which would take about $\frac{L}{\langle v \rangle}$ time, and scale by this time scale. We can expect the times we are interested in to be of the same order of magnitude as this time scale.

$$t^* = \frac{t}{\left(\frac{L}{\langle v \rangle}\right)} = \frac{\langle v \rangle t}{L}$$

Finally, for pressure, there isn't one always correct way to scale it. When inertial effects are more important (turbulent flow), we can scale by kinetic energy per unit area $\rho \langle v \rangle^2$.

$$P^* = \frac{P}{\rho \langle v \rangle^2}$$

When viscous forces dominate (creeping or laminar flow), we can scale by $\frac{\mu \langle v \rangle}{L}$.

$$P^* = \frac{P}{\frac{\mu \langle v \rangle}{L}} = \frac{PL}{\mu \langle v \rangle}$$

This term might look random, but it comes from a little manipulation of the Reynolds number $\frac{\rho \langle v \rangle L}{\mu}$, and is related to viscous effects.

$$\frac{\rho \langle v \rangle L}{\mu} = \frac{\rho \langle v \rangle^2}{\frac{\mu \langle v \rangle}{L}} = \frac{P}{\frac{\mu \langle v \rangle}{L}}$$

To start, let's consider the case where inertial terms dominate. Let's substitute our nondimensional numbers:

$$\rho \frac{\langle v \rangle}{\frac{L}{\langle v \rangle}} \frac{\delta v^*}{\delta t^*} + \rho \langle v \rangle v^* \frac{\langle v \rangle}{L} \frac{\delta v^*}{\delta x^*} = \mu \frac{\langle v \rangle}{L^2} \frac{\delta^2 v^*}{\delta x^{*2}} - \frac{\rho \langle v \rangle^2}{L} \frac{\delta P^*}{\delta x^*} + \rho g$$

$$\frac{\rho \langle v \rangle^2}{L} \frac{\delta v^*}{\delta t^*} + \frac{\rho \langle v \rangle^2}{L} v^* \frac{\delta v^*}{\delta x^*} = \frac{\mu \langle v \rangle}{L^2} \frac{\delta^2 v^*}{\delta x^{*2}} - \frac{\rho \langle v \rangle^2}{L} \frac{\delta P^*}{\delta x^*} + \rho g$$

Dividing everything by $\frac{\rho \langle v \rangle^2}{L}$:

$$\frac{\delta v^*}{\delta t^*} + v^* \frac{\delta v^*}{\delta x^*} = \frac{\mu}{\rho \langle v \rangle L} \frac{\delta^2 v^*}{\delta x^{*2}} - \frac{\delta P^*}{\delta x^*} + \frac{gL}{\langle v \rangle^2}$$

Notice that $\frac{\mu}{\rho \langle v \rangle L}$ is the inverse of the Reynolds number, which we will denote Re:

$$\frac{\delta v^*}{\delta t^*} + v^* \frac{\delta v^*}{\delta x^*} = \left(\frac{1}{Re}\right) \frac{\delta^2 v^*}{\delta x^{*2}} - \frac{\delta P^*}{\delta x^*} + \frac{gL}{\langle v \rangle^2}$$

The Reynolds number is the ratio of inertial forces to viscous forces. If inertial forces dominate, as we have stated, then the Reynolds number is very large, so its inverse is very small. Then, $\left(\frac{1}{Re}\right) \frac{\delta^2 v^*}{\delta x^{*2}}$ can be ignored. In this case, the original equation can be simplified to:

$$\rho \frac{\delta v_x}{\delta t} + \rho v_x \frac{\delta v_x}{\delta x} = -\frac{\delta P}{\delta x} + \rho g$$

Now, for the case where viscous forces dominate, we once again substitute our nondimensional numbers into the initial equation:

$$\rho \frac{\langle v \rangle}{\frac{L}{\langle v \rangle}} \frac{\delta v^*}{\delta t^*} + \rho \langle v \rangle v^* \frac{\langle v \rangle}{L} \frac{\delta v^*}{\delta x^*} = \mu \frac{\langle v \rangle}{L^2} \frac{\delta^2 v^*}{\delta x^{*2}} - \frac{\frac{\mu \langle v \rangle}{L}}{L} \frac{\delta P^*}{\delta x^*} + \rho g$$

$$\frac{\rho \langle v \rangle^2}{L} \frac{\delta v^*}{\delta t^*} + \frac{\rho \langle v \rangle^2}{L} v^* \frac{\delta v^*}{\delta x^*} = \frac{\mu \langle v \rangle}{L^2} \frac{\delta^2 v^*}{\delta x^{*2}} - \frac{\mu \langle v \rangle}{L^2} \frac{\delta P^*}{\delta x^*} + \rho g$$

Dividing everything by $\frac{\mu \langle v \rangle}{L^2}$:

$$\frac{\rho \langle v \rangle L}{\mu} \frac{\delta v^*}{\delta t^*} + \frac{\rho \langle v \rangle L}{\mu} v^* \frac{\delta v^*}{\delta x^*} = \frac{\delta^2 v^*}{\delta x^{*2}} - \frac{\delta P^*}{\delta x^*} + \frac{\rho L^2}{\mu \langle v \rangle} g$$

Notice the Reynolds number again:

$$Re \frac{\delta v^*}{\delta t^*} + Re\, v^* \frac{\delta v^*}{\delta x^*} = \frac{\delta^2 v^*}{\delta x^{*2}} - \frac{\delta P^*}{\delta x^*} + \frac{\rho L^2}{\mu \langle v \rangle} g$$

In this case, viscous forces dominate, so the Reynolds number is very small. Then, the starting equation simplifies to:

$$0 = \mu \frac{\delta^2 v_x}{\delta x^2} - \frac{\delta P}{\delta x} + \rho g$$

Fluid Dynamics

Fluids Under Shear

Broadly, fluids are liquids and gases which, as opposed to solids, cannot resist shear stress applied to them. Let's illustrate what happens when a fluid is under shear.

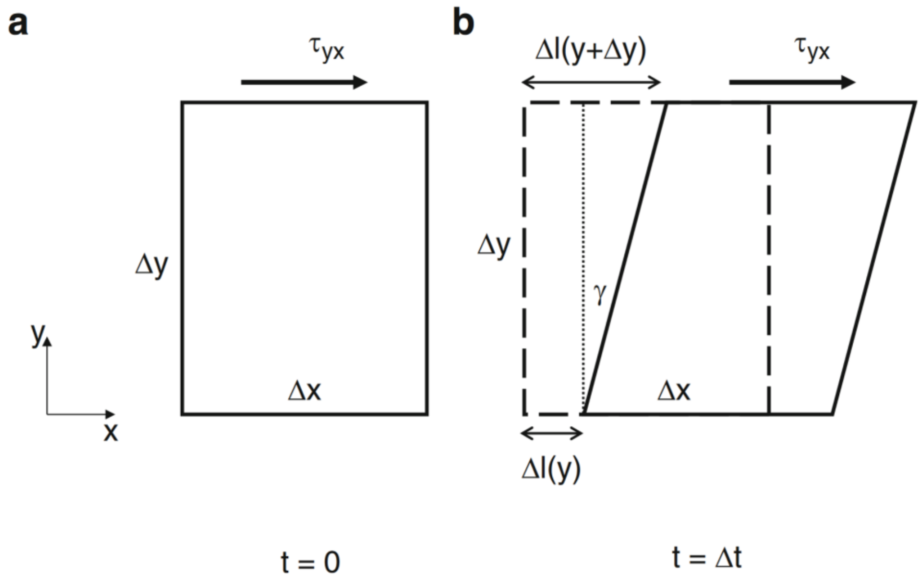


Fig. 1. Diagram of a fluid under shear [1].

Here, let's imagine a fluid is between two plates, and the top plate is pushed to the right, subjecting the fluid to a shear stress τ_{yx}. The shear stress notation τ_{yx} means that the stress is applied on the face normal to the y axis and acts in the x-direction. As a reminder, shear stress is the shearing force divided by the cross-sectional area, or the area of the top face. After a certain time Δt, this shear stress causes movement and deformation of the fluid. The fluid moves to the right by a distance of Δl, which is a function which varies with respect to y. We denote the distance at position y by $\Delta l(y)$ and the distance at position $y + \Delta y$ by $\Delta l(y + \Delta y)$.

Importantly, we can relate the shear stress to the fluid velocity. First, shear stress is proportional to the strain rate:

$$\tau = \mu \frac{d\gamma}{dt}$$

where μ is viscosity and γ is the shear strain, or the angle $\tan(\gamma) = \frac{\Delta l(y+\Delta y)-\Delta l(y)}{\Delta y}$.

For small angles, $\tan(\gamma) \cong \gamma$, so:

$$\gamma = \frac{\Delta l(y + \Delta y) - \Delta l(y)}{\Delta y}$$

Since distance is the product of velocity and time, we can replace $\Delta l(y)$ by $v_x(y)\Delta t$, where $v_x(y)$ is the x-direction velocity of the fluid at position y. We can do the same for position $y + \Delta y$.

$$\gamma = \frac{v_x(y + \Delta y)\Delta t - v_x(y)\Delta t}{\Delta y} = \left(\frac{v_x(y + \Delta y) - v_x(y)}{\Delta y}\right)\Delta t$$

If we assume that Δy is very small and take the limit as $\Delta y \to 0$, the fraction becomes a derivative:

$$\gamma = \lim_{\Delta y \to 0}\left(\frac{v_x(y + \Delta y) - v_x(y)}{\Delta y}\right)\Delta t = \frac{dv_x}{dy}\ \Delta t$$

We now have an expression for γ. Now we calculate $\frac{d\gamma}{dt}$ using the limit definition of derivative, remembering that γ at time 0 was 0.

$$\frac{d\gamma}{dt} = \frac{\gamma(t = \Delta t) - \gamma(t = 0)}{\Delta t} = \frac{\frac{dv_x}{dy}\ \Delta t - 0}{\Delta t} = \frac{dv_x}{dy}$$

So we know that $\frac{d\gamma}{dt} = \frac{dv_x}{dy}$! We can put it in the original equation to get:

$$\tau_{yx} = \mu\frac{d\gamma}{dt} = \mu\frac{dv_x}{dy}$$

What this shows is that the shear stress is directly related to the velocity gradient of a fluid. We have proven our constitutive relationship, Newton's law of viscosity.

Momentum flux and Conventions

The reason this velocity gradient develops is due to the random motion of molecules. You can imagine the fluid as being made of multiple layers stacked on top of one another. The layer of molecules in contact with the top surface will move at the same velocity as the top surface, and the layer of molecules in contact with the bottom surface will have the same velocity as the bottom surface. Let's imagine that the bottom surface and thus bottom fluid layer have no velocity, while the top surface has some velocity. Fast molecules will then diffuse through the lower layers, bringing the average velocity of the layer up. Meanwhile, slow molecules from the bottom layers will diffuse upwards, bringing the average velocity of those layers down. After a certain amount of time, this will create a stable velocity gradient, as the figure below shows.

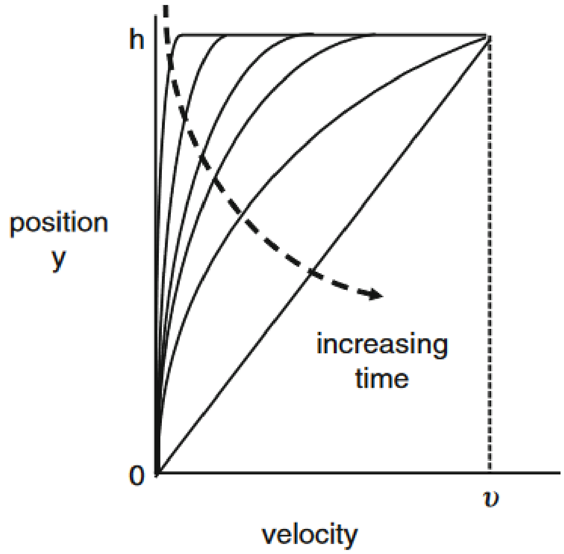


Fig. 2. Developing velocity gradient [1].

This diffusion of fast and slow molecules creates a **momentum flux**. This momentum flux is equal to the shear stress applied to the fluid, so we already have a governing equation for it:

$$momentum\ flux = \tau_{yx} = -\mu \frac{dv_x}{dy}$$

Two sign conventions exist. In this course (unless it has changed) you will use the convention **that shear stress is positive when going towards the right (positive x-direction) on the bottom face and towards the left (negative x-direction) on the top face**. If the shear stress is positive, it will create a **negative velocity gradient** $\left(\frac{dv_x}{dy} < 0\right)$ and a **positive momentum flux going towards the positive y-direction**.

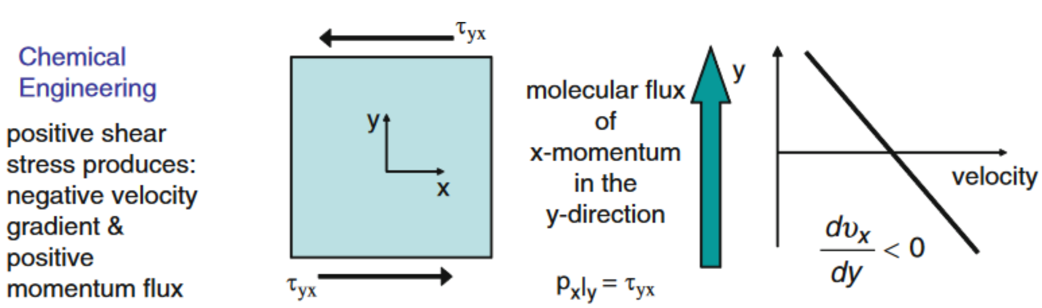


Fig. 3. Sign convention [1].

Rheology

A Newtonian fluid's viscosity is not affected by shear rate. However, not all liquids behave so nicely. For example, ketchup is shear-thinning: it becomes less viscous when higher shear stress is applied to it, for example by shaking or hitting the bottle. For non-Newtonian fluids, Newton's law of viscosity can still be used, but instead of using a constant viscosity μ, we need to use an **apparent viscosity** $\eta(\dot{\gamma})$, which is a function of the shear strain rate.

$$\tau_{yx} = -\eta(\dot{\gamma})\frac{dv_x}{dy}$$

Fluid models

Power law model

$$\tau_{yx} = -K\left|\frac{dv_x}{dy}\right|^{n-1}\frac{dv_x}{dy}, \qquad \eta = K\left|\frac{dv_x}{dy}\right|^{n-1} = K|\dot{\gamma}|^{n-1}$$

- *n*: power law index
  - depends on temperature and pressure
  - 0<*n*<1, pseudoplastic (shear thinning)
  - *n*>1, dilatant (shear thickening)
  - *n*=1, Newtonian
- *K*: flow consistency index

Bingham fluid

Bingham fluids are Newtonian fluids with a yield stress. Below the yield stress τ_y, they act as a solid, but they act as a Newtonian fluid with viscosity μ above the yield stress.

An example of a Bingham fluid is toothpaste: if you don't squeeze the tube, it doesn't flow out at all, even under the effect of gravity. This is because gravity doesn't cause enough shear to overcome the yield stress.

$$if\ |\tau_{yx}| \le \tau_y, \qquad \frac{dv_x}{dy} = 0$$

$$\tau_{yx} = \tau_y + \mu\left(-\frac{dv_x}{dy}\right), \qquad \frac{dv_x}{dy} < 0$$

$$\tau_{yx} = -\tau_y + \mu\left(-\frac{dv_x}{dy}\right), \qquad \frac{dv_x}{dy} > 0$$

Casson fluid

This model also has a yield stress but is also shear thinning.

$$if\ |\tau_{yx}| \le \tau_y, \qquad \frac{dv_x}{dy} = 0$$

$$\sqrt{\tau_{yx}} = S\sqrt{-\frac{dv_x}{dy}} + \sqrt{\tau_y}, \qquad \frac{dv_x}{dy} < 0$$

$$\sqrt{-\tau_{yx}} = S\sqrt{\frac{dv_x}{dy}} + \sqrt{\tau_y}, \qquad \frac{dv_x}{dy} > 0$$

- S is a material property with units $\sqrt{Pa \cdot s} = \sqrt{\frac{kg}{m \cdot s}}$

Herschel-Bulkley Fluid

It combines Power law fluid with a yield stress.

$$if\ |\tau_{yx}| \le \tau_y, \qquad \frac{dv_x}{dy} = 0$$

$$\tau_{yx} = \tau_y + K\left(-\frac{dv_x}{dy}\right)^n, \qquad \frac{dv_x}{dy} < 0$$

$$\tau_{yx} = -\tau_y - K\left(\frac{dv_x}{dy}\right)^n, \qquad \frac{dv_x}{dy} > 0$$

Blood Rheology

Blood viscosity depends on protein concentration, hematocrit, and vessel radius:

- Effect of proteins: Higher protein concentrations increase viscosity.
- Effect of red blood cells:
  - Higher hematocrit increases apparent viscosity
  - Higher sheer rate decreases apparent viscosity
    - This is because at high shear rates, red blood cells start deforming, while at low shear rates, red blood cells start aggregating.
  - Smaller conduit radius decreases apparent viscosity
    - This is because of the Fahraeus-Lindqvist effect.

The **Fahraeus-Lindqvist effect** is caused by an effective decrease in blood hematocrit due to the lower capillary radius. Let's take the location of a red blood cell to be its center. A red blood cell could not be located right at the wall of the capillary, because it simply wouldn't fit: the red blood cell has a certain radius which can't overlap with the capillary wall. So the region within a red blood cell radius of the capillary wall is completely free of red blood cells. This means that the hematocrit (H) of blood in the capillary can be expressed as:

$$H = \begin{cases} H_0, & r \leq R - R_c \\ 0, & r > R - R_c \end{cases}$$

where r is the radius in the capillary, H_0 is the hematocrit of free blood, and R_c is the red blood cell radius.

Since the hematocrit of the capillary is lower, the concentration of red blood cells upstream and downstream of the capillary are higher. To avoid creating a bottleneck and delivering less red blood cells downstream, the red blood cells must go faster, reducing apparent viscosity.

Constitutive law for blood

Blood follows the Casson model for low shear rates and the Newtonian model for high shear rates.

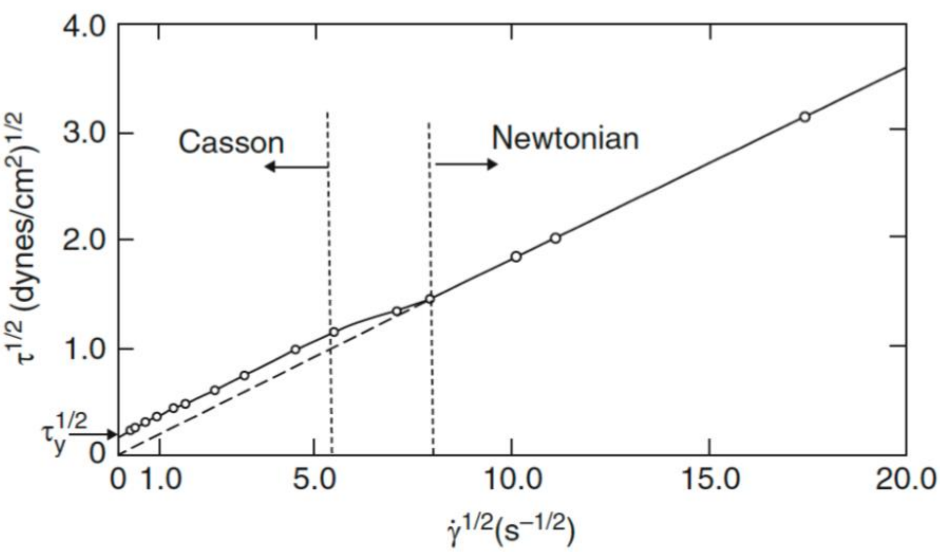


Fig. 4. Constitutive law for blood [2].

Boundary conditions

To solve differential equations, boundary conditions are needed. Here are useful boundary conditions that you should know.

Solid-liquid interface (no slip)

We assume that the fluid layer in contact with any solid adheres to it, thus has the same velocity as the solid. If the solid is immobile, so is that layer of fluid.

$$v_{x,fluid} = v_{x,solid}, \qquad at\ interface$$

Interface between immiscible liquids

At the surface between two liquids, velocity and shear stress are equal.

$$v_{x,fluid\ 1} = v_{x,fluid\ 2}, \qquad \tau_{yx,fluid\ 1} = \tau_{yx,fluid\ 2}, \qquad at\ interface$$

Liquid-gas interface

Gas does not apply any shear stress on a fluid, because the viscosity of a gas is much smaller than that of a liquid.

$$\tau_{yx} = 0, \qquad at\ interface$$

Symmetry

If the geometry and forces applied allow it, the velocity can be symmetrical around a centerline. This means that the velocity gradient is 0 at that centerline.

$$\frac{\delta v_x}{\delta y}\Big|_{x=0} = 0$$

Macroscopic Approach

When using the macroscopic approach, we generally apply conservation laws to the entire system. We don’t really care about velocity gradients, only what comes in and out of the system. We can apply conservation of mass, momentum, or energy to the system.

Conservation of mass

When to use

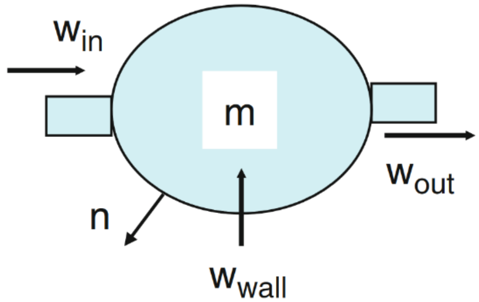
Conservation of mass is the easiest method to apply, but the most limited. It is useful when all we care about is **what goes into and out of the system**. It cannot deal with forces, pressure, or friction.

General equation

From conservation of mass, we know that:

$$\left\{\begin{matrix} Rate\ of\ accumulation\ of \\ mass\ in\ the\ system \end{matrix}\right\} = \left\{\begin{matrix} rate\ mass\ enters \\ the\ system \end{matrix}\right\} - \left\{\begin{matrix} rate\ mass\ exits \\ the\ system \end{matrix}\right\}$$

In a system, mass can only enter or exit through inlets, outlets, or by seeping through the walls if they are permeable. For now, we'll use a system with only one inlet and outlet to keep things simple. For example, a system can look like this:



$$\frac{dm}{dt} = w_{wall} + w_{in} - w_{out}$$

Fig. 5. Diagram of conservation of mass [1].

where m is mass, t is time, and w is the mass flow rate through the walls (if they are permeable) and through the inlet and outlet.

We can obtain the mass by integrating the density over the entire volume of the system:

$$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho dV$$

where ρ is the density and V is the volume of the system.

We can obtain the mass flow rates by integrating the mass flux over the surface area of the walls and inlets.

$$w_{wall} = -\int_{S} (\rho\vec{v}) \cdot \vec{n} dS$$

$$w_{in} = \int_{A_{in}} \rho_{in} v_{in} dA$$

where v is the velocity, S is the surface area of the wall, A is the surface area of the inlet, and $\vec{n}$ is the outwards unit vector normal to the surface of the wall. Note that the negative sign is present because the unit vector points towards the outside; if it pointed inwards, there would be no negative sign.

So the general equation for conservation of mass is:

$$\frac{d}{dt}\int_{V} \rho dV = -\int_{S} (\rho\vec{v}) \cdot \vec{n} dS + \int_{A_{in}} \rho_{in} v_{in} dA - \int_{A_{out}} \rho_{out} v_{out} dA$$

Assumptions

Assumption: uniform density across inlet and outlet areas.

This allows us to take the density out of the derivative.

$$w_{in} = \int_{A_{in}} \rho_{in} v_{in} dA = w_{in} = \rho_{in} \int_{A_{in}} v_{in} dA = \rho_{in} \langle v_{in} \rangle A_{in}$$

where $\langle v \rangle$ is the average velocity.

The equation then becomes:

$$\frac{d}{dt}\int_{V} \rho dV = -\int_{S} (\rho\vec{v}) \cdot \vec{n} dS + \rho_{in} \langle v_{in} \rangle A_{in} - \rho_{out} \langle v_{out} \rangle A_{out}$$

Assumption: well mixed, uniform density in the system, and uniform density across inlet and outlet areas.

This simplification allows us to say that:

$$\frac{d}{dt}\int_{V} \rho dV = \frac{d}{dt}(\rho V)$$

$$\frac{d}{dt}(\rho V) = -\int_{S} (\rho\vec{v}) \cdot \vec{n} dS + \rho_{in}\langle v_{in}\rangle A_{in} - \rho\langle v_{out}\rangle A_{out}$$

Assumption: incompressible fluid (constant density).

The equation becomes a conservation of volume equation, since we can take density out of the equation.

$$\frac{dV}{dt} = Q_{V,wall} + Q_{V,in} + Q_{V,out} = -\int_{S} \vec{v} \cdot \vec{n} dS + \langle v_{in}\rangle A_{in} - \langle v_{out}\rangle A_{out}$$

Assumption: steady state flow.

If the system is at steady state, nothing changes with time, so the left side of the equation becomes 0.

$$\frac{dm}{dt} = 0 = w_{wall} + w_{in} - w_{out}$$

Assumption: no seepage through walls (impermeable walls).

This allows you to remove the wall term, obtaining:

$$\frac{dm}{dt} = w_{in} - w_{out}$$

To keep things simple, we already made the assumption that there only was one inlet and outlet. If there are multiple inlets and/or outlets, add them together:

$$\frac{dm}{dt} = w_{wall} + \sum_{i=1}^{num.\ of\ inlets} w_{in,i} - \sum_{i=1}^{num.\ of\ outlets} w_{out,i}$$

You will typically be able to assume that there is uniform density across the inlets and outlets, and that the walls are impermeable. Then, the equation you will usually start with is:

$$\boxed{\frac{dm}{dt} = \sum_{i=1}^{num\ inlets} \rho\langle v_{in}\rangle A_{in} - \sum_{j=1}^{num\ outlets} \rho\langle v_{out}\rangle A_{out}}$$

Conservation of momentum

When to use

Conservation of momentum is most useful when dealing with **forces**. If the question asks for the forces exerted by the walls, that's a dead giveaway that you should use conservation of momentum. It can also deal with pressure and gravity. It can't really deal with friction very well, except for frictional forces.

General equation

Conservation of momentum starts similarly to conservation of energy:

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ momentum\ \ in\ the\ system \end{Bmatrix} = \begin{Bmatrix} rate\ momentum \\ enters\ the\ system \\ by\ convection \end{Bmatrix} - \begin{Bmatrix} rate\ momentum \\ exits\ the\ system \\ by\ convection \end{Bmatrix} + \begin{Bmatrix} rate\ of\ production \\ of\ momentum \end{Bmatrix}$$

Momentum is the product of mass and velocity, meaning that:

$$\vec{p} = m\vec{v} = \rho V\vec{v}$$

where p is momentum, m is mass, v is velocity, ρ is density, and V is volume.

For the rate of accumulation of momentum, we integrate the momentum per unit volume over the volume of the system:

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ momentum\ \ in\ the\ system \end{Bmatrix} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\int_V \rho\vec{v}dV$$

For the rate momentum enters or leaves the system by convection, we integrate the momentum flux over the cross-sectional area of the inlet or outlet. Momentum flux is the product of velocity and mass flux:

$$\begin{Bmatrix} rate\ momentum \\ enters\ the\ system \\ by\ convection \end{Bmatrix} = \int_A \rho v^2 dA\,\vec{e}$$

where $\vec{e}$ is the unit vector in the direction of velocity and A is the area of the inlet or outlet.

Momentum can be generated by applying an external force to the fluid. Three forces are relevant: gravity, the force exerted by the fluid on the system walls, and pressure at inlets and outlets.

$$\begin{Bmatrix} rate\ of\ production \\ of\ momentum \end{Bmatrix} = \sum\vec{F} = m\vec{g} - \vec{R} + \sum^{Num\ inlets} \int_A P_{inlet} dA\,\vec{e} - \sum^{Num\ outlets} \int_A P_{outlet} dA\,\vec{e}$$

where $\vec{R}$ is the force exerted by the fluid on the system walls (inverse the sign for the force exerted by the wall on the fluid), $\vec{g}$ is the acceleration due to gravity, and P is the pressure.

Putting everything together:

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\int_V \rho\vec{v}dV = \sum^{num\ inlets} \int_A \rho v^2 dA\,\vec{e} - \sum^{num\ outlets} \int_A \rho v^2 dA\,\vec{e} + m\vec{g} - \vec{R}$$

$$+ \sum^{Num\ inlets} \int_A P_{inlet} dA\,\vec{e} - \sum^{Num\ outlets} \int_A P_{outlet} dA\,\vec{e}$$

Assumptions

Assumption: you will almost always assume that density and pressure are uniform at entrances and exits. If density is uniform at the inlet:

$$\int_A \rho v^2 dA\,\vec{e} = \rho \int_A v^2 dA\,\vec{e} = \rho\langle v^2\rangle A\vec{e} = \rho K_2 \langle v\rangle^2 A\vec{e}$$

where $\langle v^2\rangle$ is the average square velocity and $\langle v\rangle$ is the average velocity.

Note that, in general, $\langle v^2\rangle \neq \langle v\rangle^2$. However, $\langle v^2\rangle$ is really hard to work with. This is why we use the constant K. By definition:

$$K_k = \frac{\langle v^k\rangle}{\langle v\rangle^k}$$

The value of K depends on the geometry of the conduit and the flow regime, and will usually be given in the problem. Usually, K is 1 for turbulent flows.

If pressure is uniform at the inlet:

$$\int_A P_{inlet} dA\,\vec{e} = P_{inlet} A_{inlet} \vec{e}$$

Putting everything back together, this is the equation you will usually start with:

$$\boxed{\frac{d\vec{p}}{dt} = \sum_{i=1}^{num\ inlets} \left(\rho_i K_{2i} \langle v\rangle_i^{\ 2} + P_i\right) A_i \vec{e_i} - \sum_{j=1}^{num\ outlets} \left(\rho_j K_{2j} \langle v\rangle_j^{\ 2} + P_j\right) A_j \vec{e_j} + m\vec{g} - \vec{R}}$$

Assumption: steady state:

$$\frac{d\vec{p}}{dt} = 0$$

Conservation of energy

When to use

Conservation of energy is the most involved method but is useful when **friction** has to be taken into account.

General equation

As always, we start with the conservation statement:

$$\left\{\begin{matrix} Rate\ of\ accumulation\ of \\ energy\ \ in\ the\ system \end{matrix}\right\} = \left\{\begin{matrix} rate\ energy \\ enters\ system \end{matrix}\right\} - \left\{\begin{matrix} rate\ energy \\ exits\ system \end{matrix}\right\} + \left\{\begin{matrix} rate\ of\ production \\ of\ energy \end{matrix}\right\}$$

For energy accumulation in the system, we can integrate the specific energy (energy per unit mass) over the mass of the system.

$$\left\{\begin{matrix} Rate\ of\ accumulation\ of \\ energy\ \ in\ the\ system \end{matrix}\right\} = \frac{d}{dt}\int_m \hat{\mathrm{E}}dm$$

where t is time, m is mass, and $\hat{\mathrm{E}}$ is specific energy.

Specific energy can be divided into internal energy, kinetic energy, and potential energy:

$$\hat{\mathrm{E}} = \hat{\mathrm{U}} + \widehat{K} + \widehat{\Phi}$$

$$\hat{\mathrm{U}} = c_p(T - T_R), \qquad \widehat{K} = \frac{v^2}{2}, \qquad \widehat{\Phi} = gh$$

where $\hat{\mathrm{U}}$ is specific internal energy, $\widehat{K}$ is specific kinetic energy, $\widehat{\Phi}$ is specific potential energy, c_p is specific heat capacity, T is temperature, T_R is reference temperature, v is velocity, g is gravitational acceleration, and h is height.

For energy entering and leaving the system, we have to consider both the heat added to the system through conduction and radiation, and the heat added through convection. Energy flux is equal to the product of mass flux and specific energy. We can use the surface integral of energy flux over the surface area of inlets and outlets to obtain the energy flow rate due to convection.

$$\left\{\begin{matrix} rate\ energy \\ enters\ system \end{matrix}\right\} - \left\{\begin{matrix} rate\ energy \\ exits\ system \end{matrix}\right\} = \dot{Q}_s + \sum_i^{num\ inlets} \int_{A_i} \hat{\mathrm{E}}_i \rho_i v_i dA - \sum_i^{num\ outlets} \int_{A_i} \hat{\mathrm{E}}_i \rho_i v_i dA$$

where $\dot{Q}_s$ is the heat added to the system through conduction and radiation and ρ is density.

We can also split the specific energy into its components:

$$\int_{A_i} \hat{\mathrm{E}}_i \rho_i v_i dA = \int_{A_i} \left(\hat{U}_i + \frac{v_i^2}{2} + \hat{\Phi}_i \right) \rho_i v_i dA$$

Finally, though energy cannot truly be "created", energy can be added in our system through two ways. First, heat can be generated in our system through chemical reactions, viscous heat dissipation, electrical heating, and so on; the energy was already present, but wasn't taken into account in the equation.

Second, the system can do work on its surroundings. This includes shaft work and friction work at the inlets and outlets. In addition, since work is the product of force and displacement, it follows that, if force is constant over time:

$$\frac{d}{dt}(W) = \frac{d}{dt}(F\Delta x) = F\frac{d\Delta x}{dt} = Fv = PAv$$

where W is work, F is force, Δx is displacement, and v is velocity.

This means that the pressure in the inlets and outlets exert work on the system. Thus:

$$\begin{Bmatrix} rate\ of\ production \\ of\ energy \end{Bmatrix} = \dot{Q}_{gen} - \dot{W}_s - \dot{W}_f + \sum_i^{num\ inlets} \int_{A_i} P_i v_i dA - \sum_i^{num\ outlets} \int_{A_i} P_i v_i dA$$

where $\dot{Q}_{gen}$ is the heat generated by chemical, electrical, viscous, and other phenomena, $\dot{W}_s$ is the shaft work done by the system on the environment, $\dot{W}_f$ is the frictional work at the boundary done by the system on the environment, and P is pressure.

Putting everything together:

$$\frac{dE}{dt} = \sum_i^{num\ inlets} \int_{A_i} \left(\hat{U}_i + \frac{v_i^2}{2} + \hat{\Phi}_i + \frac{P_i}{\rho} \right) \rho_i v_i dA - \sum_i^{num\ outlets} \int_{A_i} \left(\hat{U}_i + \frac{v_i^2}{2} + \hat{\Phi}_i + \frac{P_i}{\rho} \right) \rho_i v_i dA + \dot{Q}_s + \dot{Q}_{gen} - \dot{W}_s - \dot{W}_f$$

Right away, we'll make an assumption to get rid of these ugly integrals:

Assumption: uniform density, pressure, temperature, internal energy, potential energy over cross section of inlets and outlets.

$$\frac{dE}{dt} = \dot{Q}_s + \dot{Q}_{gen} - \dot{W}_s - \dot{W}_f + \sum_i^{num\ inlets} w_i \left(\hat{U}_i + \frac{\langle v_i^3 \rangle}{2\langle v_i \rangle} + \hat{\Phi}_i + \frac{P_i}{\rho} \right) - \sum_i^{num\ outlets} w_i \left(\hat{U}_i + \frac{\langle v_i^3 \rangle}{2\langle v_i \rangle} + \hat{\Phi}_i + \frac{P_i}{\rho} \right)$$

where w is mass flux.

We can also use the K (remember $K_i = \frac{\langle v^i \rangle}{\langle v \rangle^i}$) to simplify the equation:

$$\frac{dE}{dt} = \dot{Q}_s + \dot{Q}_{gen} - \dot{W}_s - \dot{W}_f + \sum_i^{num\ inlets} w_i \left(\hat{U}_i + \frac{K_{3i} \langle v_i \rangle^2}{2} + \hat{\Phi}_i + \frac{P_i}{\rho} \right) - \sum_i^{num\ outlets} w_i \left(\hat{U}_i + \frac{K_{3i} \langle v_i \rangle^2}{2} + \hat{\Phi}_i + \frac{P_i}{\rho} \right)$$

Engineering Bernoulli

Through a series of assumptions, we can come to an "Engineering Bernoulli" equation. We will make these assumptions often, so it is useful to have an equation directly.

Assumption: isothermal, incompressible fluid, no chemical reactions (or radioactive decay, electrical heating, etc.) or heat through surfaces. Also, internal/potential energy and pressure are uniform across cross-section, as before.

This allows us to remove the internal energy $\hat{U}$ and heat added through conduction $\dot{Q}_s$ terms. In addition, the $\dot{Q}_{gen}$ term now only represents viscous heat dissipation, or friction loss, and will be replaced by E_V.

We obtain:

$$\frac{dE}{dt} = -\dot{W}_s - E_V + \sum_i^{num\ inlets} w_i \left(\frac{K_{3i} \langle v_i \rangle^2}{2} + \hat{\Phi}_i + \frac{P_i}{\rho} \right) - \sum_i^{num\ outlets} w_i \left(\frac{K_{3i} \langle v_i \rangle^2}{2} + \hat{\Phi}_i + \frac{P_i}{\rho} \right)$$

where E_V is the rate at which mechanical energy is converted to heat by viscous dissipation, or friction loss.

Assumption: steady state ($\frac{d}{dt} = 0$), single inlet and outlet, in addition to the previous assumptions.

Note that, by conservation of mass, if there is a single inlet and outlet, the mass flux at the inlet and outlet must be equal.

$$\boxed{\frac{\dot{W}_s}{w} + \frac{E_V}{w} = \frac{1}{2} \left(K_{3in} \langle v_{in} \rangle^2 - K_{3out} \langle v_{out} \rangle^2 \right) + g(h_{in} - h_{out}) + \frac{(P_{in} - P_{out})}{\rho}}$$

This is the engineering Bernoulli equation. We can further simplify it into the "normal" Bernoulli equation you might already know by ignoring viscous dissipation and shaft work.

Assumption: no shaft work, negligible friction (inviscid fluid with negligible friction), in addition to all previous assumptions

In this case, since the fluid is inviscid, thus has a viscosity of zero, the velocity profile will be flat, so K_3 will be 1. In addition, no viscous dissipation will occur, so E_V will be 0.

$$\frac{\langle v_{in}\rangle^2}{2} + gh_{in} + \frac{P_{in}}{\rho} = \frac{\langle v_{out}\rangle^2}{2} + gh_{out} + \frac{P_{out}}{\rho}$$

The Bernoulli equation has limited use because of the inviscid fluid assumption. You will most often use the engineering Bernoulli equation, in which the assumptions are more reasonable. Still, to use the equation, we need to be able to find this E_V term.

Friction loss

The frictional force can be expressed as:

$$F_k = fKA = f\frac{1}{2}(\rho\langle v\rangle^2)(P_w L)$$

where F_k is the frictional force, f is a unitless friction factor, K is kinetic energy/volume, A is the characteristic area which is the surface area in contact with the fluid, ρ is density, v is velocity, P_w is wetted perimeter (perimeter in contact with fluid), and L is the length of the conduit.

To find the frictional force, let's use conservation of momentum, but with only one inlet and outlet of the same size and at steady-state in the conduit below.

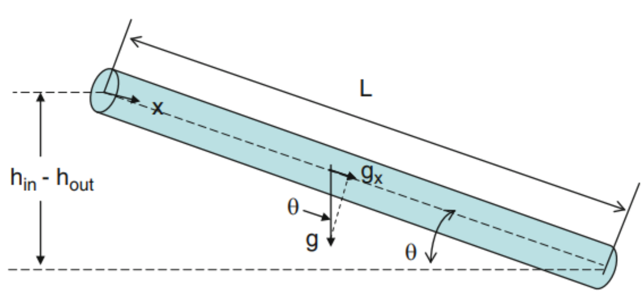


Fig. 6. Diagram of a cylindrical conduit with one inlet and one outlet [1].

Because of these assumptions, from conservation of mass, we know that the average velocity at the entrance and exit must be the same. Thus, using conservation of momentum at steady state, we only need to take pressure, gravity, and frictional force into account:

$$\frac{d\vec{p}}{dt} = 0 = P_{in}A_c - P_{out}A_c + mg\sin\theta - R$$

$$R = F_k = (P_{in} - P_{out})A_c + \rho A_c L \sin\theta\, g = (P_{in} - P_{out})A_c + \rho A_c(h_{in} - h_{out})g$$

$$F_k = A_c\big((P_{in} - P_{out}) + \rho(h_{in} - h_{out})g\big)$$

where p is momentum, t is time, P is pressure, A_c is cross-sectional area of the conduit, m is total fluid mass, g is gravitational acceleration, θ is the angle of the conduit, R is the force applied by the walls on the fluid which is the same as the frictional force F_k, L is the length of the conduit, h is the height of the inlet and outlet, and ρ is density.

Now we use the engineering Bernoulli equation in this same situation:

$$\frac{E_V}{w} = g(h_{in} - h_{out}) + \frac{(P_{in} - P_{out})}{\rho}$$

where E_V is the rate at which energy is converted to heat by viscous dissipation, or friction loss, and w is the mass flow rate.

Notice how similar it is to the equation we obtained for F_k using conservation of momentum. In fact, we can divide F_k by ρA_c to find $\frac{E_V}{w}$.

$$\frac{F_k}{\rho A_c} = \frac{(P_{in} - P_{out})}{\rho} + (h_{in} - h_{out})g = \frac{E_V}{w}$$

We can conclude that:

$$\frac{E_V}{w} = \frac{F_k}{\rho A_c} = \frac{f\,\frac{1}{2}(\langle v\rangle^2)(P_w L)}{A_c}$$

To simplify this, we can use the hydraulic diameter:

$$D_h = \frac{4A_c}{P_w}$$

where D_h is the hydraulic diameter, A_c is the cross-sectional area, and P_w is the wetted perimeter. If the conduit is circular, then the hydraulic diameter is simply the diameter of the pipe.

Replacing into the equation:

$$\frac{E_v}{w} = 4f\frac{L}{D_h}\left(\frac{1}{2}\langle v\rangle^2\right)$$

Now we need to find this friction factor f. First, however, we must determine if we can ignore entry effects. Typically, there is a region close to the entrance where the velocity profile has yet to develop, called the hydrodynamic entrance region. This may cause additional friction loss. This effect is negligible for long enough conduits. In addition, it is usually very short and thus negligible if flow is turbulent.

In order to ignore entry effects, in laminar flow:

$$\frac{L}{d} > 0.2Re_d$$

where L is the conduit length, d is the conduit diameter, and Re_d is the Reynolds number with the diameter as the characteristic length. The entrance length can also be calculated as:

$$\frac{L_e}{d} = [(0.619)^{1.6} + (0.0567Re_d)^{1.6}]^{\frac{1}{1.6}}$$

where L_e is the entrance length.

The friction factor depends on the conduit geometry and Reynolds number. Depending on the case, you might have to use different tables.

Case: laminar flow in a circular conduit, entrance effects can be ignored

In this case, the friction factor can simply be calculated as:

$$f = \frac{16}{Re_d}$$

Case: laminar flow in a circular conduit, entrance effects cannot be ignored

Use the following graph:

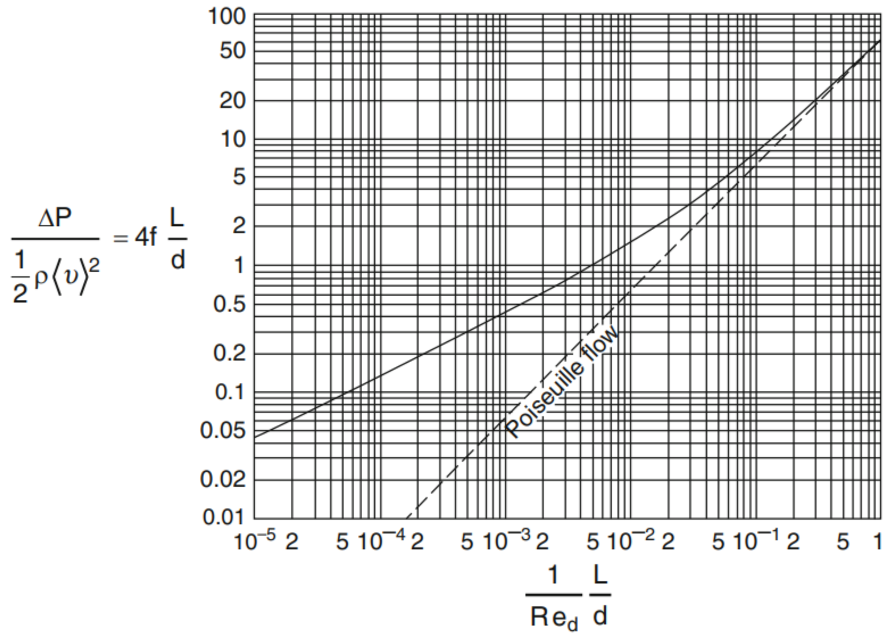


Fig. 7. Friction loss in circular tubes with entry effects [1].

"Poiseuille flow" represents the case in which entrance effects can be ignored.

Case: nonlaminar/turbulent flow in a circular conduit

In this case, the roughness of the conduit will be important. Simply use the Moody diagram.

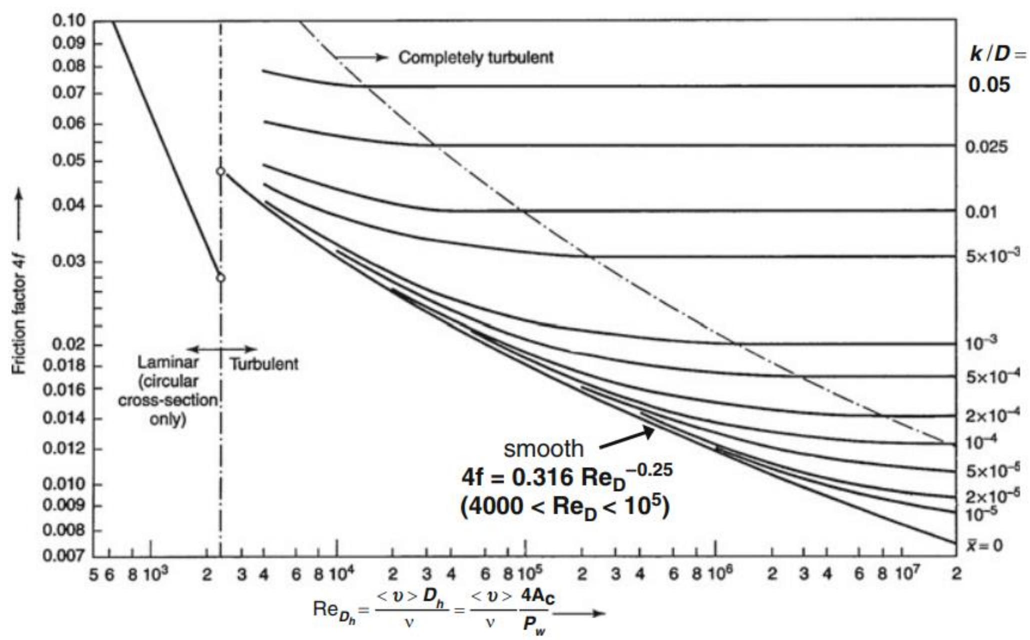


Fig. 8. Moody diagram [1].

Case: laminar flow in a noncircular conduit

In this case:

$$E_V = \frac{12\mu L A_c \langle v \rangle}{\rho B d^3 M_0}$$

where μ is viscosity, B and d are geometric factors, and M_0 is a dimentionless coefficient obtained from the graph below.

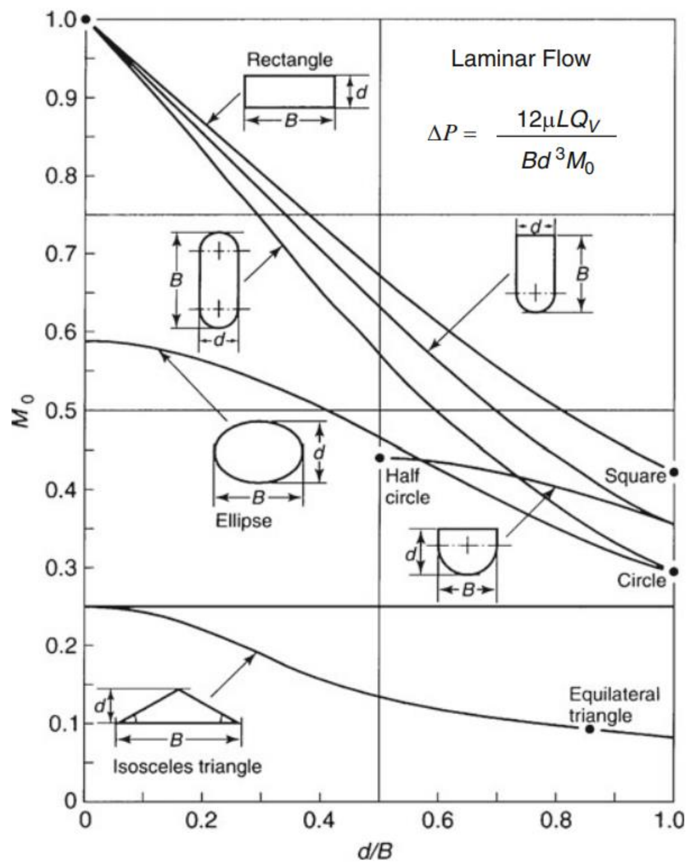


Fig. 9. Values of M_0 for different conduit geometries [1].

Alternatively, the resistance to flow can be used, which is calculated as:

$$R_f = \frac{12\mu L}{B d^3 M_0}$$

where R_f is the resistance to flow through the conduit. If the conduit is circular, then

$$R_f = \frac{8\mu L}{\pi R^4}$$

where R is the conduit radius.

This resistance can be used to easily find the volumetric flow rate through a conduit:

$$Q_V = \frac{\Delta P}{R_f}$$

More information in the electrical circuit analogy section.

To close the topic of friction loss, we must consider fittings, which can be expansions or contractions of the conduit, elbow joints, orifices, junctions, etc. These introduce additional friction loss. We use a friction factor K_w:

$$K_w = \frac{E_V}{w\frac{1}{2}\langle v\rangle^2}$$

The velocity is always taken **downstream** of the fitting. This friction factor can be obtained from tables. We then add up all the friction loss terms from all the conduits and fittings to obtain the total friction loss.

$$\frac{E_V}{w} = \sum_{i=1}^{num\ fittings} K_{w,i}\frac{1}{2}\langle v_{out,i}\rangle^2 + \sum_{j=1}^{num\ conduits} 4f_j\frac{L_j}{D_{h,j}}\frac{1}{2}\langle v_j\rangle^2$$

External flow

Fluid movement past a stationary object creates drag. This drag force has two components. Frictional drag is caused by the shear stress applied by the fluid on the object due to the velocity gradient of the fluid (think of Newton's law of viscosity). Form drag is caused by pressure differences creating a net force acting on the object. The pressure difference can be between the region upstream and downstream of the object, or it can be between the top and bottom of the object, in which case the object will lift.

To derive the drag equation, we can use the frictional force equation again:

$$F_k = fKA = \frac{1}{2}f\rho v^2 A$$

where F_k is the frictional force, f is a unitless friction factor, K is kinetic energy/volume, A is the characteristic area, ρ is fluid density, and v is velocity.

Since we are dealing with external flow, the relevant velocity and area will change. The relevant area will be the frontal area of the object, or the cross-sectional area of the object when looking at it in the direction of fluid flow. Also, we will take the velocity to be the fluid velocity far away from the object. Finally, the friction factor becomes a drag coefficient C_D. The force of drag then becomes:

$$F_D = \frac{1}{2} C_D \rho v_\infty^2 A_c$$

where F_D is the drag force, C_D is the drag coefficient, v_∞ is the velocity of the fluid far away from the object, and A_c is the cross-sectional area of the object.

In the case of **laminar flow** past a stationary **sphere**, the drag coefficient is:

$$C_D = \frac{24}{Re_{D_s}} = \frac{24\mu}{\rho v_\infty D_s}$$

where D_s is the diameter of the sphere and μ is fluid viscosity.

Replacing into the original equation, we obtain Stokes' law:

$$F_D = \frac{1}{2}\left(\frac{24\mu}{\rho v_\infty D_s}\right)\rho v_\infty^2 \left(\frac{\pi D_s}{4}\right) = 3\pi\mu D_s v_\infty$$

$$F_D = 3\pi\mu D_s v_\infty$$

If the sphere is moving, you can simply take the difference between sphere and fluid velocity.

$$F_D = -3\pi\mu D_s (v_s - v_\infty)$$

Compliance

So far, we've dealt with rigid conduits, but some conduits (like blood vessels) can **stretch or collapse depending on the pressures** applied to them – a concept called compliance. To quantify this, let's first define the transmural pressure as the difference between internal pressure inside the conduit and external pressure:

$$P_{tm} = P - P_e$$

where P_{tm} is transmural pressure, P is internal pressure, and P_e is external pressure.

In general, when transmural pressure is positive, the cross-sectional area of the conduit increases, while when it is negative, the cross-sectional area of the conduit decreases (or collapses). We can then define the **compliance** of the vessel as the rate of vessel volume change depending on transmural pressure:

$$C = \frac{dV}{dP_{tm}}$$

where C is compliance and V is vessel volume.

Let's relate this to conservation of mass. We know that if fluid density is constant, conservation of mass basically becomes a conservation of volume:

$$\frac{dV}{dt} = \sum_{i=1}^{num\ inlets} Q_{V,i} - \sum_{j=1}^{num\ outlets} Q_{V,j}$$

where Q_V is the volumetric flow rate at the inlets and outlets and t is time. Note that we are ignoring any flow through the walls.

Now, if we treat the derivative as a fraction, and using the definition of compliance:

$$\frac{dV}{dt} = \frac{dV}{dP_{tm}}\frac{dP_{tm}}{dt} = C\frac{dP_{tm}}{dt} = C\left(\frac{dP}{dt} - \frac{dP_e}{dt}\right)$$

We can combine the two equations:

$$\frac{dV}{dt} = C\frac{dP_{tm}}{dt} = \sum_{i=1}^{num\ inlets} Q_{V,i} - \sum_{j=1}^{num\ outlets} Q_{V,j}$$

$$C\left(\frac{dP}{dt} - \frac{dP_e}{dt}\right) = \sum_{i=1}^{num\ inlets} Q_{V,i} - \sum_{j=1}^{num\ outlets} Q_{V,j}$$

which allows us to relate compliance, internal and external pressures, and flow rates.

Electrical Circuit Analogy

As stated in the introduction, the movement of charges in an electrical circuit is part of transport phenomena. In fact, an electrical circuit and a network of pipes can look very similar and, mathematically, behave very similarly.

In an electrical circuit, charge is transported through electrical wires, creating an electrical current, driven by a difference in electrical potential. This leads to the following equation:

$$I = \frac{\Delta V}{R}$$

where I is current, ΔV is the electrical potential difference, and R is electrical resistance.

We can relate this to fluid flow. In this case, fluid is transported through a conduit, creating a volumetric flow rate, driven by a difference in pressure. This leads to the following equation:

$$Q_V = \frac{\Delta P}{R_f}$$

where Q_V is the volumetric flow rate, ΔP is pressure difference, and R_f is resistance to flow.

The equation holds as long as entry effects can be ignored. In electrical circuits, the wire resistance is usually ignored, but the resistance to flow of conduits is usually significant. As seen previously, for laminar flow in circular conduits, the resistance can be calculated as:

$$R_f = \frac{8\mu L}{\pi R^4}$$

where μ is fluid viscosity, L is the conduit length, and R is the conduit radius.

For noncircular conduits, the equation is:

$$R_f = \frac{12\mu L}{Bd^3 M_0}$$

where B and d are dimensions of the conduit and M_0 is obtained from **Fig. 9**.

This means that conduits in parallel and series can be treated as resistors in parallel and series:

Parallel	Series
$\frac{1}{R_t} = \sum_{i=1}^{num\ conduits} \frac{1}{R_{f,i}}$	$R_t = \sum_{i=1}^{num\ conduits} R_{f,i}$

where R_t is the total resistance.

This can be used to easily deal with complex networks of conduits.

Example

A device is composed of a 50 cm long, 10 cm diameter circular pipe which splits into ten 50 cm long, 1 cm diameter circular pipes. The pressure difference between the inlet and outlets of the device is 100 Pa. What is the total volumetric flowrate of water passing through the device if it has constant density 1000 kg/L and a viscosity of 0.001 Pa s. Ignore the friction at fittings and bifurcations, and any entry effects.

We can treat this as a circuit looking like this:

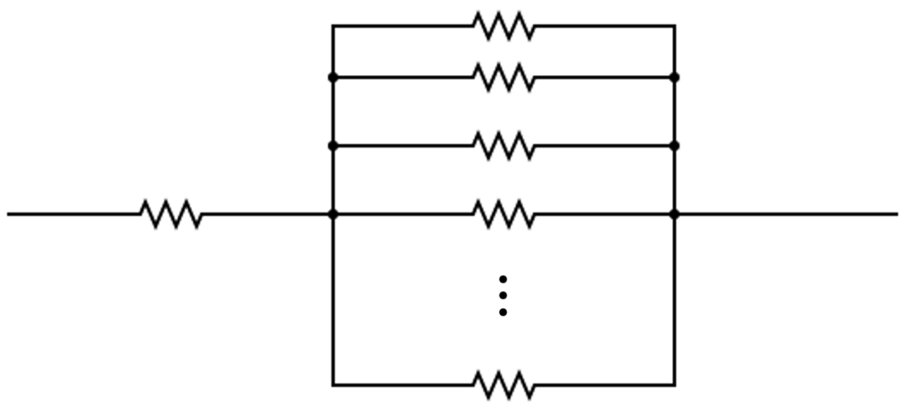


Fig. 10. Equivalent circuit with resistances in series and in parallel.

The resistance in the 10 cm diameter pipe is:

$$R_f = \frac{8\mu L}{\pi R^4} = 204\frac{Pas}{m^3}$$

The resistance in one of the 1 cm diameter pipes is:

$$R_f = 2040\frac{Pas}{m^3}$$

Then, the total resistance is:

$$R_t = 204\frac{Pas}{m^3} + \frac{1}{10\left(\frac{1}{2040\frac{Pas}{m^3}}\right)} = 408\frac{Pas}{m^3}$$

Then, we can find the volumetric flow rate:

$$Q_V = \frac{\Delta P}{R_f} = \frac{100Pa}{408\frac{Pas}{m^3}} = 0.245\frac{m^3}{s}$$

Shell Balance

While the macroscopic approach is good enough to look at what goes into and out of the system, it can't deal with spatial variations inside the system itself. For example, it can't be used to obtain the velocity profile of a fluid. On the other hand, the method of shell balance is well suited to dealing with 1 dimensional steady-state flow problems where **spatial variations are important**. It is usually used when the shear stress distribution, velocity distribution, or flow rate are needed.

The basic idea is similar to the idea for a derivative: we take a very small shell, such that everything is uniform within this shell, and use conservation of mass and momentum on this shell. Then, we take the limit as the volume of the shell approaches 0. This will give us derivatives with respect to x or y for shear stress or pressure, from which we can obtain the velocity gradient using a constitutive relationship.

General method

1. Define a shell.
    - The shell is a small region of the fluid of interest. When working is cartesian coordinates, it is a small rectangular prism, while it should be a hollow cylinder (or prism with an annular base) when working with cylindrical coordinates.
2. Perform a mass balance.
    - Using conservation of mass, list mass entering and leaving shell.
    - Divide by the shell volume and take the limit as the shell volume goes to zero.
3. Perform a momentum balance.
    - Use conservation of momentum, list the momentum entering and leaving by convection and diffusion, and momentum created through forces.
    - Divide by the shell volume and take the limit as the shell volume approaches 0.
    - Do this for both x-direction and y-direction momentum (or r- and z- direction if working in cylindrical coordinates).
    - This will give the derivative of shear stress with respect to some direction.
4. Obtain shear stress and/or pressure profiles
    - Integrate the derivatives.
    - Apply boundary conditions if applicable.
5. Apply constitutive relationship

- Use Newton's law of viscosity to obtain the velocity profile from the shear stress profile.
- Integrate.
- Apply boundary conditions.

6. Calculate flow rate if necessary
    - The flow rate can be calculated as $Q_v = \langle v_x \rangle A_c$, where Q_v is the volumetric flow rate, $\langle v_x \rangle$ is the average velocity, and A_c is the cross-sectional area. The average velocity can be obtained by integrating the velocity profile.

Mass Balance

Let's consider a shell like the one below. Its lower left edge is located at length x and height y along the conduit, and it has a length of Δx, a height of Δy, and a width of W.

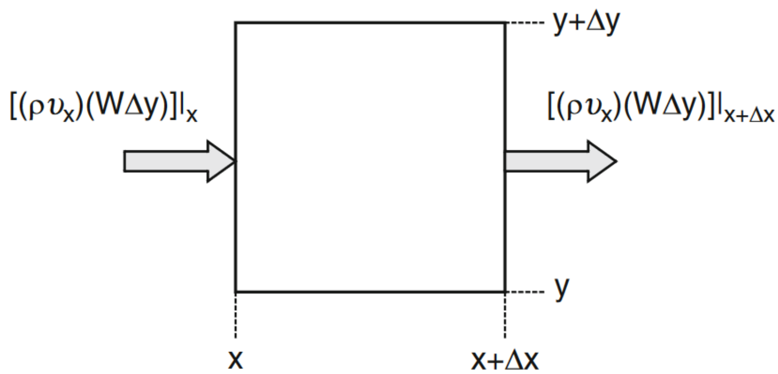


Fig. 11. Diagram of mass inflow and outflow of a shell [1].

Starting with the general conservation of mass statement:

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ mass\ in\ the\ system \end{Bmatrix} = \begin{Bmatrix} rate\ mass\ enters \\ the\ system \end{Bmatrix} - \begin{Bmatrix} rate\ mass\ exits \\ the\ system \end{Bmatrix}$$

We are only interested in steady-state problems, so there is no accumulation of mass in the system. In addition, we are only interested in 1D problems, so we only need to consider the mass flow rate entering through the left and exiting through the right, thus the mass flow rate at position x and that at position $x + \Delta x$. The mass flow rate can be calculated as the mass flux multiplied by the area, and since the shell is very small, we can assume that density, velocity, and everything else is uniform in the shell, so there is no need for a surface integral.

$$0 = [(\rho v_x)W\Delta y]|_x - [(\rho v_x)W\Delta y]|_{x+\Delta x}$$

where ρ is density and v_x is velocity in the x-direction. Mass flux is calculated as ρv_x. The symbol $|_x$ denotes "at position x" (and $|_{x+\Delta x}$ "at position $x + \Delta$x").

We know that both W and Δy don't depend on x, so we can rearrange the equation like so:

$$0 = (\rho v_x|_x - \rho v_x|_{x+\Delta x})W\Delta y$$

We couldn't do the same for ρ or v_x, because they might vary with respect to x. Dividing by the total shell area $\Delta x\Delta yW$:

$$0 = \frac{(\rho v_x|_x - \rho v_x|_{x+\Delta x})W\Delta y}{\Delta x\Delta yW} = \frac{(\rho v_x|_x - \rho v_x|_{x+\Delta x})}{\Delta x}$$

Let's take the limit as Δx and Δy approach 0:

$$0 = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{(\rho v_x|_x - \rho v_x|_{x+\Delta x})}{\Delta x} = \lim_{\Delta x\to 0} -\frac{(\rho v_x|_{x+\Delta x} - \rho v_x|_x)}{\Delta x}$$

Notice that this is the exact same as the limit definition of a derivative:

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

So we can replace the limit with the derivative:

$$0 = -\frac{\delta(\rho v_x)}{\delta x}$$

If density is constant, then the result is simply:

$$\frac{\delta v_x}{\delta x} = 0$$

So, at steady state, in 1D problems, if density is constant, velocity is constant in the x-direction (but not necessarily in the y-direction), which is very convenient.

Momentum Balance

Momentum balance starts the same way as mass balance. Let's once again consider our shell of length Δx, height Δy, and width W, with its lower edge at position (x, y).

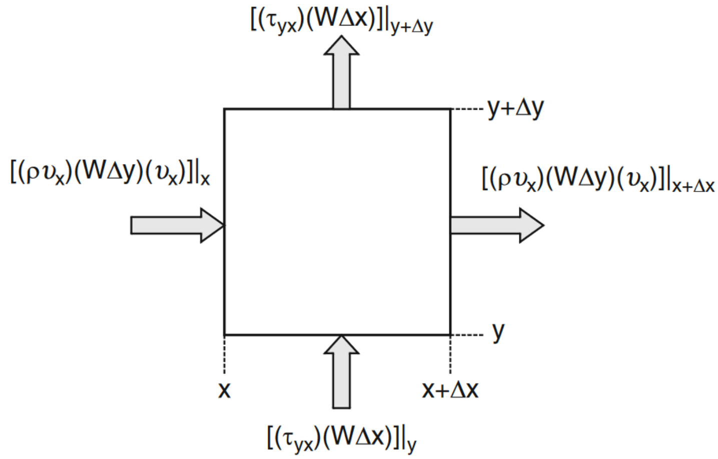


Fig. 12. Diagram of x-momentum inflow and outflow of a shell [1].

The conservation statement is:

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ momentum\ \ in\ the\ system \end{Bmatrix} = \begin{Bmatrix} rate\ momentum \\ enters\ system \end{Bmatrix} - \begin{Bmatrix} rate\ momentum \\ leaves\ system \end{Bmatrix} + \begin{Bmatrix} rate\ of\ production \\ of\ momentum \end{Bmatrix}$$

And we will, again, assume steady state:

$$0 = \begin{Bmatrix} rate\ momentum \\ enters\ system \end{Bmatrix} - \begin{Bmatrix} rate\ momentum \\ leaves\ system \end{Bmatrix} + \begin{Bmatrix} rate\ of\ production \\ of\ momentum \end{Bmatrix}$$

Momentum can generally enter or leave the system in two ways: convection and diffusion.

Fluid entering or leaving the system through the left and right faces of the shell carries its momentum with it. We then obtain momentum flow rate by multiplying momentum flux and area:

$$(\rho v_x)(v_x)(W\Delta y)$$

where ρ is density and v_x is the x-direction velocity.

On the other hand, we have already discussed the fact that shear stress at the top and bottom of the shell is equivalent to moment flux due to diffusion. Another way of thinking about this is the fact that, by multiplying the shear stress by the top or bottom area, we obtain a force, and forces generate momentum. Thus, there is additional momentum flow rate from the shear stress:

$$\tau_{yx}(W\Delta x)$$

where τ_{yx} is shear stress.

Finally, there are generally two types of forces which generate momentum: pressure and gravity.

The pressure force is the product of pressure and area. It applies on all 4 faces of the shell:

$$P(W\Delta y) \ \ or \ \ P(W\Delta x)$$

where P is pressure.

On the other hand, gravitational force is the product of mass and gravitational acceleration:

$$\rho(W\Delta x\Delta y)g$$

where g is gravitational acceleration.

Putting it all together, we can perform the momentum balance for x-direction and y-direction momentum in the shell.

y-momentum

For y-direction momentum, since flow is only in the x-direction, there is no convection or diffusion, only momentum generated through pressure and gravity.

$$0 = P(W\Delta x)|_y - P(W\Delta x)|_{y+\Delta y} - \rho(W\Delta x\Delta y)g\cos\alpha$$

where α is the angle of the conduit relative to the horizontal (0° for a perfectly horizontal conduit, 90° for a perfectly vertical conduit).

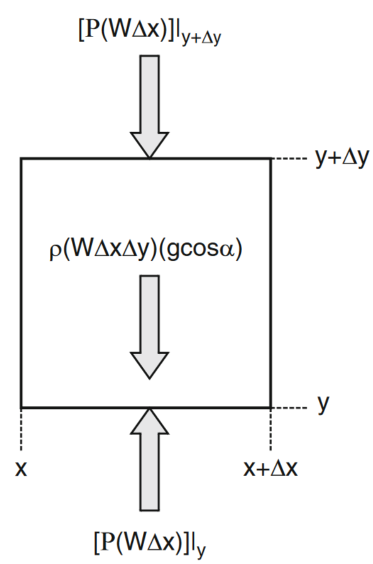


Fig. 13. Y-momentum generated through gravity and pressure [1].

Dividing by shell volume:

$$0 = \frac{P(W\Delta x)|_y - P(W\Delta x)|_{y+\Delta y} - \rho(W\Delta x\Delta y)g\cos\alpha}{W\Delta x\Delta y}$$

$$0 = \frac{P|_y - P|_{y+\Delta y}}{\Delta y} - \rho g\cos\alpha$$

Taking the limit:

$$0 = \left(\lim_{\Delta y\to 0}\frac{P|_y - P|_{y+\Delta y}}{\Delta y}\right) - \rho g\cos\alpha$$

$$0 = -\frac{\delta P}{\delta y} - \rho g\cos\alpha$$

$$\frac{\delta P}{\delta y} = -\rho g\cos\alpha$$

Integrating:

$$P = -(\rho g\cos\alpha)y + f(x)$$

We do not yet know what this $f(x)$ is, but we will find it with x-momentum balance.

x-momentum

For x-momentum, convective and diffusive fluxes, as well as forces on the left and right wall apply:

$$\begin{aligned}0 = {} & (\rho v_x)(v_x)(W\Delta y)|_x + \tau_{yx}(W\Delta x)|_y + P(W\Delta y)|_x + \rho(W\Delta x\Delta y)g\sin\alpha \\ & - (\rho v_x)(v_x)(W\Delta y)|_{x+\Delta x} - \tau_{yx}(W\Delta x)|_{y+\Delta y} - P(W\Delta y)|_{x+\Delta x}\end{aligned}$$

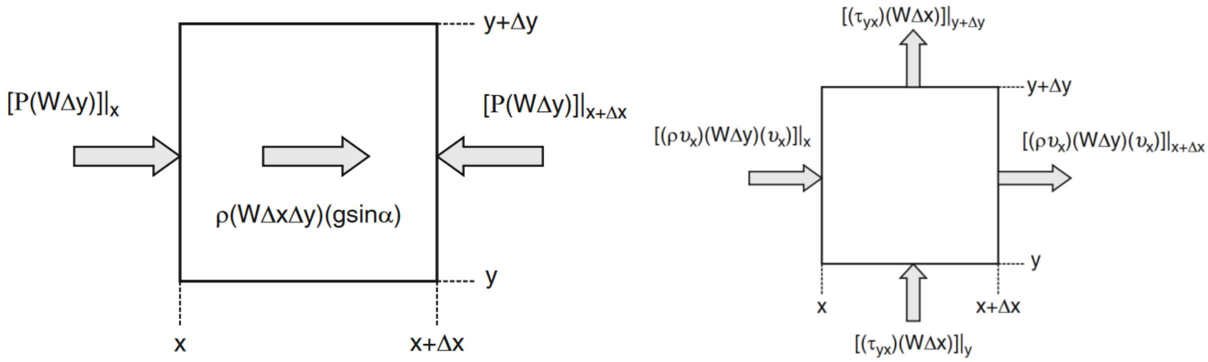


Fig. 14. Diagrams showing x-momentum generation (left), and inflow and outflow (right) of a shell. Adapted from [1].

Dividing by shell volume:

$$0 = \frac{\rho v_x^2|_x - \rho v_x^2|_{x+\Delta x}}{\Delta x} + \frac{\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y}}{\Delta y} + \frac{P|_x - P|_{x+\Delta x}}{\Delta x} + \rho g \sin \alpha$$

Taking the limit:

$$0 = -\frac{\delta(\rho v_x^2)}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta P}{\delta x} + \rho g \sin \alpha$$

Note that here, if density is constant, we found that $\frac{\delta v_x}{\delta x} = 0$ with the mass balance, meaning that velocity does not change with x. Thus:

$$\frac{\delta(\rho v_x^2)}{\delta x} = \rho \frac{\delta(v_x^2)}{\delta x} = 0$$

From here on, we need boundary conditions which depend on the problem.

Example: Flow between inclined parallel plates with pressure gradient

A fluid flows through two parallel plates at steady state. The height between the parallel plates is much smaller than the width of the parallel plates, such that we can treat it as a 1-dimensional flow problem. The plates have a length L, a height h, and are inclined at an angle α. There is a pressure difference between the two ends of the plates. The pressure at the top end is P_0 and the pressure at the bottom end is P_L. The acceleration due to gravity is g. The fluid is a Newtonian incompressible fluid with constant density ρ and viscosity μ. Find the velocity profile.

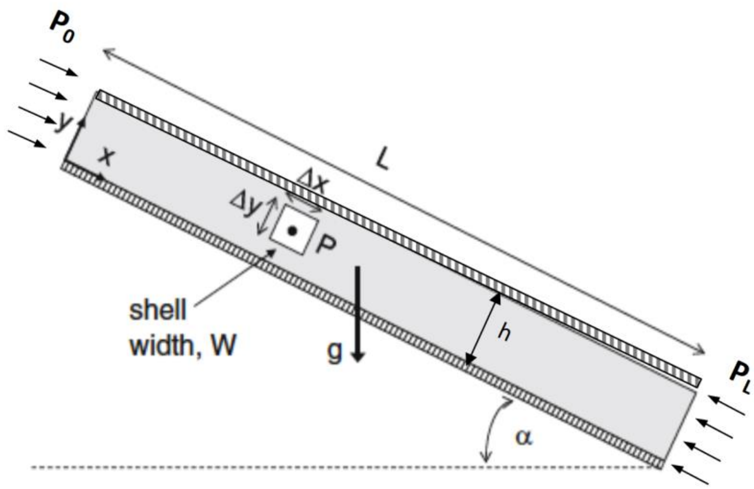


Fig. 15. Diagram of inclined parallel plates with pressure difference [3].

Though you would be expected to do the entire shell balance procedure in exercises and exams, we've already done most of the work above. We will adapt the results we've obtained in the above sections to this situation to arrive at an answer.

From the mass balance, since the fluid is incompressible, we've obtained:

$$\frac{\delta v_x}{\delta x} = 0$$

where v_x is the x-direction velocity of the fluid.

From the y-momentum balance, we've obtained:

$$P = -(\rho g \cos \alpha) y + f(x)$$

for which we've yet to find $f(x)$.

From the x-momentum balance, we've obtained:

$$0 = -\frac{\delta \rho v_x^2}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta P}{\delta x} + \rho g \sin \alpha$$

But since the fluid is incompressible, we know that:

$$\frac{\delta(\rho v_x^2)}{\delta x} = \rho \frac{\delta(v_x^2)}{\delta x} = \rho(2v_x)\frac{\delta v_x}{\delta x} = 0$$

So the x-momentum balance simplifies to:

$$\frac{\delta \tau_{yx}}{\delta y} = -\frac{\delta P}{\delta x} + \rho g \sin \alpha$$

Let's focus on the pressure for a moment. If we start from $P = -(\rho g \cos\alpha)y + f(x)$ and take the derivative with respect to x:

$$\frac{\delta P}{\delta x} = \frac{df(x)}{dx}$$

Notice that $\frac{\delta P}{\delta x}$ is only a function of x.

Now, we know from Newton's law of viscosity that:

$$\tau_{yx} = -\mu \frac{\delta v_x}{\delta y}$$

But we also know that v_x does not depend on x, and only varies with y. This means that τ_{yx}, and by extension $\frac{\delta \tau_{yx}}{\delta y}$, are only functions of y.

If we come back to $\frac{\delta \tau_{yx}}{\delta y} = -\frac{\delta P}{\delta x} + \rho g \sin\alpha$, we now realize that the left side of the equation only depends on y, and the right side only depends on x. The equality can only hold if both sides are equal to some constant.

$$\frac{d\tau_{yx}}{dy} = -\frac{\mathrm{df(x)}}{dx} + \rho g \sin\alpha = C_1$$

We can use this to solve for pressure:

$$\frac{\mathrm{df(x)}}{dx} = \rho g \sin\alpha - C_1$$

$$\mathrm{f(x)} = (\rho g \sin\alpha)x - C_1 x + C_2$$

$$P = -(\rho g \cos\alpha)y + (\rho g \sin\alpha)x - C_1 x + C_2$$

Our boundary conditions for pressure are that at $x = 0$, the pressure is P_0, and at $x = L$, the pressure is P_L. Solving for C_1 and C_2 yields:

$$C_2 = P_0$$

$$C_1 = \frac{P_0 - P_L}{L} + \rho g \sin\alpha$$

$$P(x, y) = -(\rho g \cos\alpha)y - \frac{P_0 - P_L}{L}x + P_0$$

The pressure isn't really necessary to find the velocity profile, but C_1 is very useful for finding our shear stress:

$$\frac{d\tau_{yx}}{dy} = C_1 = \frac{P_0 - P_L}{L} + \rho g \sin\alpha$$

$$\tau_{yx} = \left(\frac{P_0 - P_L}{L} + \rho g \sin\alpha\right) y + C_3$$

We don't have any boundary conditions for shear stress, since our boundary conditions are no slip at both plates ($v_x = 0$ at $x = 0$ and $x = h$). This means we're forced to carry the constant to our constitutive equation. Since the fluid is Newtonian:

$$\tau_{yx} = -\mu \frac{\delta v_x}{\delta y}$$

$$\frac{\delta v_x}{\delta y} = -\frac{\tau_{yx}}{\mu} = -\frac{1}{\mu}\left(\frac{P_0 - P_L}{L} + \rho g \sin\alpha\right) y - \frac{C_3}{\mu}$$

$$v_x = -\frac{1}{\mu}\left(\frac{P_0 - P_L}{L} + \rho g \sin\alpha\right)\frac{y^2}{2} - \frac{C_3}{\mu} y + C_4$$

Using our boundary conditions ($v_x(x = 0) = 0$ and $v_x(x = h) = 0$) we find that:

$$C_4 = 0$$

$$C_3 = -\left(\frac{P_0 - P_L}{L} + \rho g \sin\alpha\right)\frac{h}{2}$$

$$v_x = -\frac{1}{\mu}\left(\frac{P_0 - P_L}{L} + \rho g \sin\alpha\right)\frac{y^2}{2} + \frac{1}{\mu}\left(\frac{P_0 - P_L}{L} + \rho g \sin\alpha\right)\frac{h}{2} y$$

$$v_x = \frac{h^2}{2\mu}\left(\frac{P_0 - P_L}{L} + \rho g \sin\alpha\right)\left(\frac{y}{h} - \frac{y^2}{h^2}\right)$$

Note that we could have used a symmetry boundary condition $\frac{\delta v_x}{\delta y} = 0$ at $y = \frac{h}{2}$ to find C_3 before integrating $\frac{\delta v_x}{\delta y}$.

Cylindrical Shell

When conduits are circular, it makes more sense to use a cylindrical coordinate system. The shell should look like a hollow cylinder:

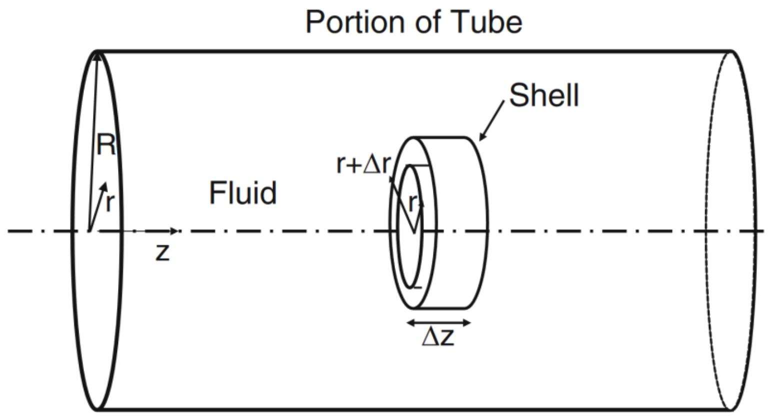


Fig. 16. Cylindrical shell [1].

Let's first calculate the shell volume. We know that the shell volume will be the difference between the volume of the outer cylinder and the inner cylinder:

$$V = \pi(r + \Delta r)^2\Delta z - \pi r^2\Delta z = \pi(2r\Delta r + \Delta r^2)\Delta z$$

However, remember that we are assuming that our shell is very small, so Δr and Δz are very small. Squaring a small number makes it even smaller, so we can say that $\Delta r^2 \ll 2r\Delta r$ because $\Delta r \ll 2r$. This allows us to simplify the volume equation to:

$$V = 2\pi r\Delta r\Delta z$$

From the same argument, the area of the faces to the right and left of the shell are:

$$A = 2\pi r\Delta r$$

Notice that **the volume of the shell depends on $\boldsymbol{r}$**. This makes sense: a shell with a larger radius would be bigger. This wasn't the case in cartesian coordinates: the shell volume didn't depend on its position. You have to be careful when doing your momentum balance, in particular for the shear stress (and radial pressure, if it applies). The shear stress balance should look like:

$$(2\pi r\Delta z)\tau_{rz}|_r - (2\pi r\Delta z)\tau_{rz}|_{r+\Delta r}$$

You might be tempted to factor out r, but you can't, because r depends on r. So you can only simplify it as:

$$\big((r\tau_{rz})|_r - (r\tau_{rz})|_{r+\Delta r}\big)2\pi\Delta z$$

When dividing by volume, this becomes:

$$\frac{\left((r\tau_{rz})|_r - (r\tau_{rz})|_{r+\Delta r}\right)2\pi\Delta z}{2\pi\Delta z\Delta r} = \frac{(r\tau_{rz})|_r - (r\tau_{rz})|_{r+\Delta r}}{r\Delta r}$$

And when taking the limit:

$$\lim_{\Delta r\to 0}\frac{(r\tau_{rz})|_r - (r\tau_{rz})|_{r+\Delta r}}{r\Delta r} = \frac{1}{r}\lim_{\Delta r\to 0}\frac{(r\tau_{rz})|_r - (r\tau_{rz})|_{r+\Delta r}}{\Delta r} = -\frac{1}{r}\frac{\delta(r\tau_{rz})}{\delta r}$$

General Method: Navier-Stokes Equations

For 3-dimensional problems and unsteady state problems in which internal variation are important, you have no choice but to use the general method, the Navier-Stokes equations. We'll first derive the Navier-Stokes equations and then look at how to use them.

Derivation

Shell Balance

As before, we'll take a small shell and apply conservation of mass and momentum, then divide by volume and take the limit as the shell volume approaches 0. However, this time, we aren't assuming 1-dimensional flow or steady state. The shell will be a small rectangular prism located at position (x, y, z) with sides of length Δx, Δy, Δz.

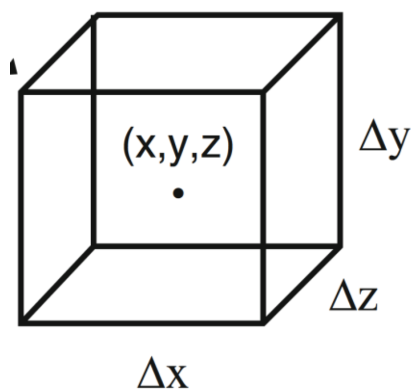


Fig. 17. Rectangular shell [1].

Conservation of mass

As before, we can write the conservation of mass statement as:

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ mass\ in\ the\ system \end{Bmatrix} = \begin{Bmatrix} rate\ mass\ enters \\ the\ system \end{Bmatrix} - \begin{Bmatrix} rate\ mass\ exits \\ the\ system \end{Bmatrix}$$

For the rate of accumulation of mass:

$$\frac{\delta m}{\delta t} = \frac{\delta}{\delta t}(\rho V) = V\frac{\delta \rho}{\delta t}$$

where m is mass, t is time, ρ is density, and V is the shell volume. Note that we can take the volume out of the derivative because the volume is constant over time.

For the rate mass enters and leaves the system, we have to take the mass flow rate at each of the 6 faces of the shell. Remember that the mass flow rate is the product of mass flux and area, and that the mass flux is the product of density and velocity.

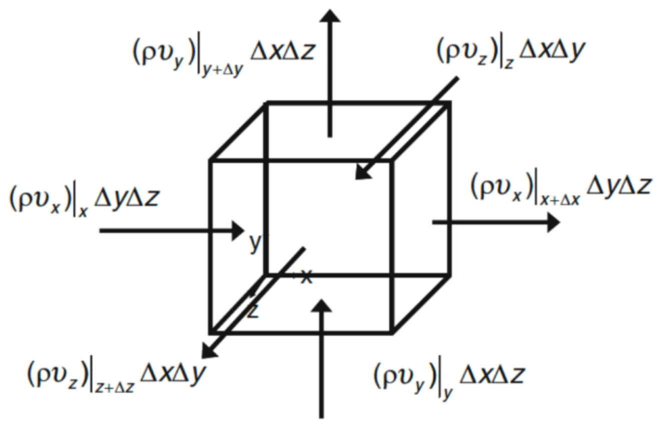


Fig. 18. Mass balance of a rectangular shell [1].

$$\begin{Bmatrix} rate\ mass\ enters \\ the\ system \end{Bmatrix} - \begin{Bmatrix} rate\ mass\ exits \\ the\ system \end{Bmatrix}$$
$$= (\rho v_x)|_x \Delta y \Delta z - (\rho v_x)|_{x+\Delta x} \Delta y \Delta z + (\rho v_y)|_y \Delta x \Delta z - (\rho v_y)|_{y+\Delta y} \Delta x \Delta z + (\rho v_z)|_z \Delta x \Delta y - (\rho v_z)|_{z+\Delta z} \Delta x \Delta y$$

where v_i is the fluid velocity in i-direction.

So, putting it all together:

$$V\frac{\delta \rho}{\delta t} = (\rho v_x)|_x \Delta y \Delta z - (\rho v_x)|_{x+\Delta x} \Delta y \Delta z + (\rho v_y)|_y \Delta x \Delta z - (\rho v_y)|_{y+\Delta y} \Delta x \Delta z + (\rho v_z)|_z \Delta x \Delta y - (\rho v_z)|_{z+\Delta z} \Delta x \Delta y$$

Dividing by the volume $V = \Delta x \Delta y \Delta z$:

$$\frac{V}{V}\frac{\delta \rho}{\delta t} = \frac{(\rho v_x)|_x \Delta y \Delta z - (\rho v_x)|_{x+\Delta x} \Delta y \Delta z}{\Delta x \Delta y \Delta z} + \frac{(\rho v_y)|_y \Delta x \Delta z - (\rho v_y)|_{y+\Delta y} \Delta x \Delta z}{\Delta x \Delta y \Delta z} + \frac{(\rho v_z)|_z \Delta x \Delta y - (\rho v_z)|_{z+\Delta z} \Delta x \Delta y}{\Delta x \Delta y \Delta z}$$

$$\frac{\delta\rho}{\delta t} = \frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x} + \frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y} + \frac{(\rho v_z)|_z - (\rho v_z)|_{z+\Delta z}}{\Delta z}$$

Taking the limit as Δx, Δy, Δz approach 0:

$$\frac{\delta\rho}{\delta t} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \left(\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x} + \frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y} + \frac{(\rho v_z)|_z - (\rho v_z)|_{z+\Delta z}}{\Delta z} \right)$$

$$\frac{\delta\rho}{\delta t} = -\frac{\delta}{\delta x}(\rho v_x) - \frac{\delta}{\delta y}(\rho v_y) - \frac{\delta}{\delta z}(\rho v_z)$$

This result is the continuity equation, which can also be expressed as:

$$\frac{\delta\rho}{\delta t} + \frac{\delta}{\delta x}(\rho v_x) + \frac{\delta}{\delta y}(\rho v_y) + \frac{\delta}{\delta z}(\rho v_z) = 0, \qquad \frac{\delta\rho}{\delta t} + \nabla \cdot (\rho\vec{v}) = 0$$

Conservation of momentum

As before, we start with the conservation statement:

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ momentum\ \ in\ the\ system \end{Bmatrix} = \begin{Bmatrix} rate\ momentum \\ enters\ the\ system \\ by\ convection \end{Bmatrix} - \begin{Bmatrix} rate\ momentum \\ exits\ the\ system \\ by\ convection \end{Bmatrix} + \begin{Bmatrix} rate\ of\ production \\ of\ momentum \end{Bmatrix}$$

Since we aren't assuming a 1-dimensional problem, we need to perform the conservation of momentum for all three dimensions. Let's focus on x-momentum for now. For the accumulation of momentum in the system:

$$\frac{\delta}{\delta t}(m v_x) = \frac{\delta}{\delta t}(\rho V v_x) = V\frac{\delta}{\delta t}(\rho v_x)$$

Again, we can take the volume out of the derivative since the shell volume does not depend on time.

As before, momentum can enter or exit the shell both through bulk fluid motion and through diffusion caused by shear stress acting on the faces. Starting with bulk fluid motion, we must consider the momentum flow rate through each of the 6 faces.

$$\begin{aligned} \begin{Bmatrix} convective\ momentum \\ transport \end{Bmatrix} \\ &= (\rho v_x v_x|_x - \rho v_x v_x|_{x+\Delta x})\Delta y \Delta z + (\rho v_y v_x|_y - \rho v_y v_x|_{y+\Delta y})\Delta x \Delta z \\ &+ (\rho v_z v_x|_z - \rho v_z v_x|_{z+\Delta z})\Delta x \Delta y \end{aligned}$$

Because the fluid could have both y-, x-, and z-velocity components, it is possible that fluid would be coming from the bottom face at an angle, like in the figure below. In this case, x-momentum would be added at a rate proportional to the y-velocity, which is why terms like $v_y v_x$ appear in the equation.

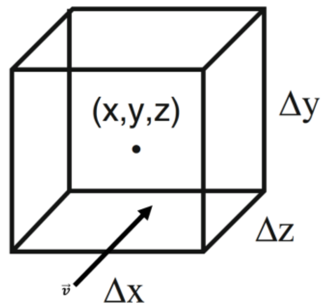


Fig. 19. Diagram of fluid entering the bottom face of the shell at an angle. In this case, x-momentum is added to the shell at a rate proportional to y-velocity and vice-versa [1].

Next, for shear stress, we need to consider the shear acting at each face. Remember that the shear stress notation τ_{ij} means that the stress is applied on the face normal to the i axis and acts in the j-direction. Since we care about x-momentum, we only care about x-direction shear stress for now, so all our shear stress will be of the form τ_{ix}.

$$\begin{Bmatrix} diffusive\ momentum \\ transport \end{Bmatrix}$$
$$= (\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x})\Delta y\Delta z + (\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y})\Delta x\Delta z$$
$$+ (\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z})\Delta x\Delta y$$

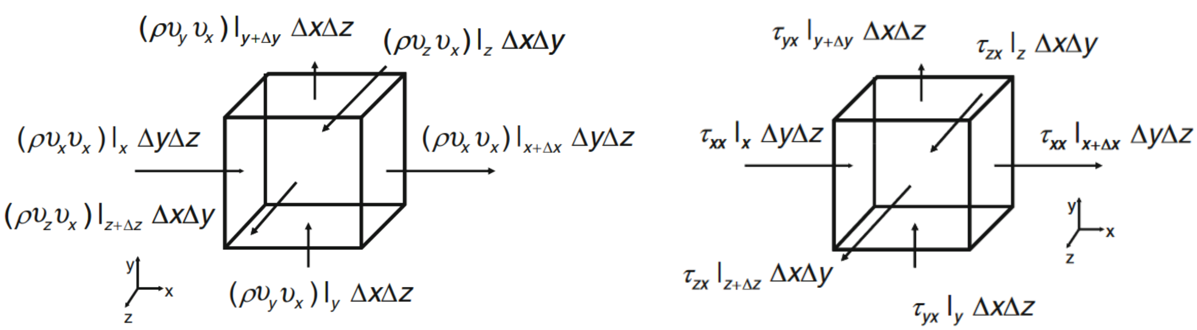


Fig. 20. X-momentum entering and leaving shell by convective and diffusive transport. Adapted from [1].

Finally, we need to take into account forces generating momentum, namely pressure and gravity. Since we care about x-momentum, we only care about the pressure acting in the x-direction, so only pressure acting on the left and right faces. Though it is counterintuitive, we also need to consider x-direction gravity, which might or might not exist depending on how you defined the coordinate system.

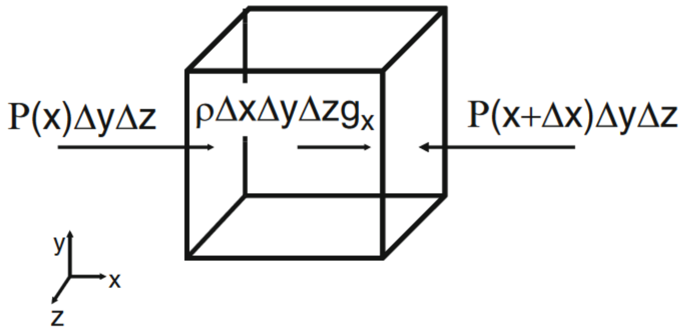


Fig. 21. X-momentum generated by forces of gravity and pressure [1].

$$\begin{Bmatrix} momentum \\ generated \end{Bmatrix} = (P|_x - P|_{x+\Delta x})\Delta y\Delta z + \rho g_x \Delta x\Delta y\Delta z$$

Putting it all together:

$$\begin{aligned} V\frac{\delta}{\delta t}(\rho v_x) = {} & (\rho v_x v_x|_x - \rho v_x v_x|_{x+\Delta x})\Delta y\Delta z + \left(\rho v_y v_x|_y - \rho v_y v_x|_{y+\Delta y}\right)\Delta x\Delta z \\ & + (\rho v_z v_x|_z - \rho v_z v_x|_{z+\Delta z})\Delta x\Delta y + (\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x})\Delta y\Delta z \\ & + \left(\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y}\right)\Delta x\Delta z + (\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z})\Delta x\Delta y + (P|_x - P|_{x+\Delta x})\Delta y\Delta z \\ & + \rho g_x \Delta x\Delta y\Delta z \end{aligned}$$

Dividing by the shell volume $V = \Delta x\Delta y\Delta z$:

$$\begin{aligned} \frac{\delta}{\delta t}(\rho v_x) = {} & \frac{(\rho v_x v_x|_x - \rho v_x v_x|_{x+\Delta x})}{\Delta x} + \frac{\left(\rho v_y v_x|_y - \rho v_y v_x|_{y+\Delta y}\right)}{\Delta y} + \frac{(\rho v_z v_x|_z - \rho v_z v_x|_{z+\Delta z})}{\Delta z} \\ & + \frac{(\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x})}{\Delta x} + \frac{\left(\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y}\right)}{\Delta y} + \frac{(\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z})}{\Delta z} \\ & + \frac{(P|_x - P|_{x+\Delta x})}{\Delta x} + \rho g_x \end{aligned}$$

Taking the limit as Δx, Δy, Δz approach 0:

$$\frac{\delta}{\delta t}(\rho v_x) = -\frac{\delta}{\delta x}(\rho v_x v_x) - \frac{\delta}{\delta y}\left(\rho v_y v_x\right) - \frac{\delta}{\delta z}(\rho v_z v_x) - \frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} - \frac{\delta P}{\delta x} + \rho g_x$$

Now, we can simplify this equation using the product rule $\left(\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right)$:

$$\begin{aligned} \rho\frac{\delta v_x}{\delta t} + v_x\frac{\delta \rho}{\delta t} = {} & -\rho v_x\frac{\delta v_x}{\delta x} - v_x\frac{\delta}{\delta x}(\rho v_x) - \rho v_y\frac{\delta v_x}{\delta y} - v_x\frac{\delta}{\delta y}\left(\rho v_y\right) - \rho v_z\frac{\delta v_x}{\delta z} - v_x\frac{\delta}{\delta z}(\rho v_z) \\ & - \frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} - \frac{\delta P}{\delta x} + \rho g_x \end{aligned}$$

$$\rho \frac{\delta v_x}{\delta t} + \left(v_x \frac{\delta \rho}{\delta t} + v_x \frac{\delta}{\delta x}(\rho v_x) + v_x \frac{\delta}{\delta y}(\rho v_y) + v_x \frac{\delta}{\delta z}(\rho v_z) \right)$$
$$= -\rho v_x \frac{\delta v_x}{\delta x} - \rho v_y \frac{\delta v_x}{\delta y} - \rho v_z \frac{\delta v_x}{\delta z} - \frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} - \frac{\delta P}{\delta x} + \rho g_x$$

$$\rho \frac{\delta v_x}{\delta t} + v_x \left(\frac{\delta \rho}{\delta t} + \frac{\delta}{\delta x}(\rho v_x) + \frac{\delta}{\delta y}(\rho v_y) + \frac{\delta}{\delta z}(\rho v_z) \right)$$
$$= -\rho v_x \frac{\delta v_x}{\delta x} - \rho v_y \frac{\delta v_x}{\delta y} - \rho v_z \frac{\delta v_x}{\delta z} - \frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} - \frac{\delta P}{\delta x} + \rho g_x$$

Recall from the continuity equation from conservation of mass that:

$$\frac{\delta \rho}{\delta t} + \frac{\delta}{\delta x}(\rho v_x) + \frac{\delta}{\delta y}(\rho v_y) + \frac{\delta}{\delta z}(\rho v_z) = 0$$

The equation simplifies to:

$$\rho \frac{\delta v_x}{\delta t} = -\rho v_x \frac{\delta v_x}{\delta x} - \rho v_y \frac{\delta v_x}{\delta y} - \rho v_z \frac{\delta v_x}{\delta z} - \frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} - \frac{\delta P}{\delta x} + \rho g_x$$

As stated before, this is for x-momentum only. We would have to repeat this for y- and z-momentum to obtain the result:

$$\rho \frac{\delta v_x}{\delta t} = -\rho v_x \frac{\delta v_x}{\delta x} - \rho v_y \frac{\delta v_x}{\delta y} - \rho v_z \frac{\delta v_x}{\delta z} - \frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} - \frac{\delta P}{\delta x} + \rho g_x$$

$$\rho \frac{\delta v_y}{\delta t} = -\rho v_x \frac{\delta v_y}{\delta x} - \rho v_y \frac{\delta v_y}{\delta y} - \rho v_z \frac{\delta v_y}{\delta z} - \frac{\delta \tau_{xy}}{\delta x} - \frac{\delta \tau_{yy}}{\delta y} - \frac{\delta \tau_{zy}}{\delta z} - \frac{\delta P}{\delta y} + \rho g_y$$

$$\rho \frac{\delta v_z}{\delta t} = -\rho v_x \frac{\delta v_z}{\delta x} - \rho v_y \frac{\delta v_z}{\delta y} - \rho v_z \frac{\delta v_z}{\delta z} - \frac{\delta \tau_{xz}}{\delta x} - \frac{\delta \tau_{yz}}{\delta y} - \frac{\delta \tau_{zz}}{\delta z} - \frac{\delta P}{\delta z} + \rho g_z$$

These are a form of the Cauchy momentum equation and, along with the continuity equation, form the equations of motion. Notice that, so far, **we have made no assumptions**, so this equation can be used in general. However, the continuity equation and these three equations have a total of 13 unknowns (x-, y-, z-velocity, the 9 stresses, and pressure) for 4 equations. We need to simplify them a little to arrive to Navier-Stokes.

Torque Balance

The shear stresses acting on the shell will introduce torque. For now, let's work in 2D. Let's consider a shell at position (x, y) with sides of length dx and dy, and let's take the depth to be 1.

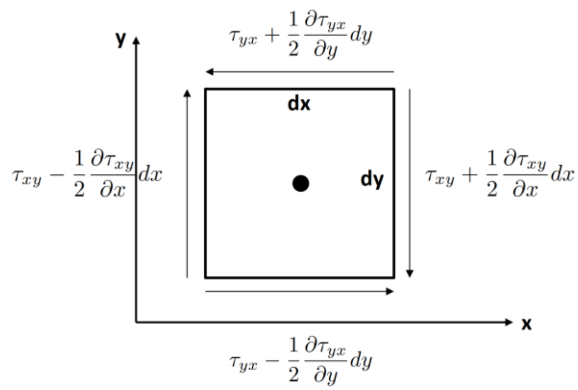


Fig. 22. Moments acting on the sides of the shell [1].

We know the value of the shear stresses τ_{xy}, τ_{xy} at position (x, y), that is at the center, but we do not know their values at the sides of the shell, so at positions $x \pm \frac{dx}{2}$ and $y \pm \frac{dy}{2}$. To obtain those, we can use the tailor expansion:

$$f(x) = f(a) + (x-a)\frac{df}{dx}(a) + \frac{(x-a)^2}{2!}\frac{d^2 f}{dx^2}(a) + \cdots$$

In this case, we ignore the derivatives of order higher than 1, as they should be very small. So for shear stress at the walls:

$$\tau_{xy}\big|_{x \pm \frac{dx}{2}} = \tau_{xy}|_x + \left(\pm\frac{dx}{2}\right)\frac{\delta \tau_{xy}}{\delta x}|_x$$

$$\tau_{yx}\big|_{y \pm \frac{dy}{2}} = \tau_{yx}|_y + \left(\pm\frac{dy}{2}\right)\frac{\delta \tau_{yx}}{\delta y}|_y$$

To obtain the torque, we multiply force and distance from the center. To obtain force, we multiply shear stress by area (which is either dx or dy, since the depth is 1). So, for example, the torque created by the shear stress at the top face would be:

$$\left(\left(\tau_{yx} + \frac{dy}{2}\frac{\delta \tau_{yx}}{\delta y}\right)dx\right)\frac{dy}{2}$$

where $\tau_{yx} + \frac{dy}{2}\frac{d\tau_{yx}}{dy}$ is the shear stress at position $y + \frac{dy}{2}$, dx is the area of the face, $\left(\tau_{yx} + \frac{dy}{2}\frac{d\tau_{yx}}{dy}\right)dx$ is the force created by the shear stress, and $\frac{dy}{2}$ is the distance from the center.

We have to add the other 3 shear stresses to obtain the total torque. Keep in mind that torque is positive when acting counterclockwise:

$$total\ torque = \left(\tau_{yx} + \frac{dy}{2}\frac{\delta\tau_{yx}}{\delta y}\right)dx\frac{dy}{2} + \left(\tau_{yx} - \frac{dy}{2}\frac{\delta\tau_{yx}}{\delta y}\right)dx\frac{dy}{2} - \left(\tau_{xy} + \frac{dx}{2}\frac{\delta\tau_{xy}}{\delta x}\right)dy\frac{dx}{2} - \left(\tau_{xy} - \frac{dx}{2}\frac{\delta\tau_{xy}}{\delta x}\right)dy\frac{dx}{2}$$

We can rearrange and simplify this:

$$total\ torque = \frac{\tau_{yx}dxdy}{2} + \frac{dxdy^2}{4}\frac{\delta\tau_{yx}}{\delta y} + \frac{\tau_{yx}dxdy}{2} - \frac{dxdy^2}{4}\frac{\delta\tau_{yx}}{\delta y} - \frac{\tau_{xy}dxdy}{2} - \frac{dydx^2}{4}\frac{\delta\tau_{xy}}{\delta x} - \frac{\tau_{xy}dxdy}{2} + \frac{dydx^2}{4}\frac{\delta\tau_{xy}}{\delta x}$$

$$total\ torque = \tau_{yx}dxdy - \tau_{xy}dxdy = \left(\tau_{yx} - \tau_{xy}\right)dxdy$$

The total torque is equal to the product of angular acceleration and moment of inertia.

$$total\ torque = I\alpha$$

where I is the moment of inertia and α is angular acceleration.

For our rectangular prism, the moment of inertia is:

$$I = \frac{1}{12}m(dx^2 + dy^2)$$

where m is the mass of the shell.

This means that:

$$total\ torque = \frac{1}{12}m(dx^2 + dy^2)\alpha$$

We can set both total torque equations equal to each other:

$$total\ torque = \left(\tau_{yx} - \tau_{xy}\right)dxdy = \frac{1}{12}m(dx^2 + dy^2)\alpha$$

As always, we divide by the shell volume $V = 1 \times dxdy$:

$$\frac{\left(\tau_{yx} - \tau_{xy}\right)dxdy}{dxdy} = \frac{1}{12}\frac{m}{V}(dx^2 + dy^2)\alpha$$

$$\left(\tau_{yx} - \tau_{xy}\right) = \frac{1}{12}\rho(dx^2 + dy^2)\alpha$$

When taking the limit as dx and dy approach 0, the entire right side becomes 0:

$$\left(\tau_{yx} - \tau_{xy}\right) = \lim_{\substack{dx \to 0 \\ dy \to 0}} \frac{1}{12} \rho (dx^2 + dy^2) \alpha = 0$$

Meaning that:

$$\tau_{yx} = \tau_{xy}$$

We can repeat this in the other planes to find that:

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

We got rid of 3 of the 9 stress variables, leaving 6. Unfortunately, in the general case, that is as far as we can simplify the equation. From here on out, we need to make assumptions. We can start by applying a constitutive relationship to our fluid to simplify it further.

Navier-Stokes

Let's start by looking at deformation in the xy plane.

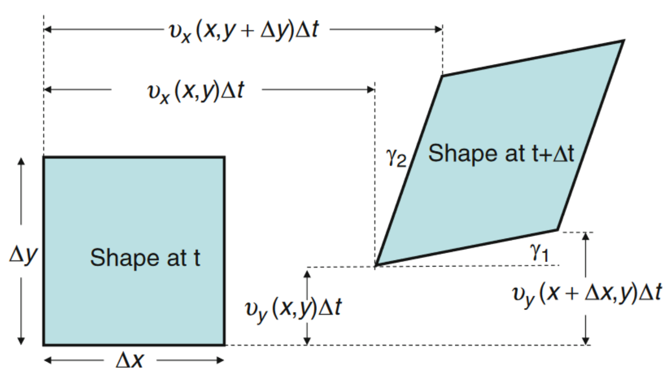


Fig. 23. 2-dimentional deformation of fluid when subjected to shear stress [1].

We've already seen in the rheology portion that, when there is deformation in 1-dimension, the rate of change of the angle γ was related to the velocity gradient.

$$\frac{d\gamma}{dt} = \frac{dv_x}{dy}$$

In this case, we are working with 2 deformations, and thus 2 shear strain rates at once, but the equation still holds:

$$\frac{d\gamma_2}{dt} = \frac{\delta v_x}{\delta y}$$

$$\frac{d\gamma_1}{dt} = \frac{\delta v_y}{\delta x}$$

This allows us to calculate the average rate of deformation, which is simply the average of the shear strain rates:

$$D_{xy} = D_{yx} = \frac{\frac{d\gamma_1}{dt} + \frac{d\gamma_2}{dt}}{2} = \frac{1}{2}\left(\frac{\delta v_x}{\delta y} + \frac{\delta v_y}{\delta x}\right)$$

If we do the same in the other planes, it gives us similar results:

$$D_{yz} = D_{zy} = \frac{1}{2}\left(\frac{\delta v_y}{\delta z} + \frac{\delta v_z}{\delta y}\right)$$

$$D_{xz} = D_{zx} = \frac{1}{2}\left(\frac{\delta v_x}{\delta z} + \frac{\delta v_z}{\delta x}\right)$$

However, recall that we also had shear stresses of the form τ_{xx} in the equation. These are shear stresses acting in the x-direction on the plane normal to the x-direction which … isn't really a shear stress, but a normal stress. Then let's look at how the normal strain is related to the velocity gradient.

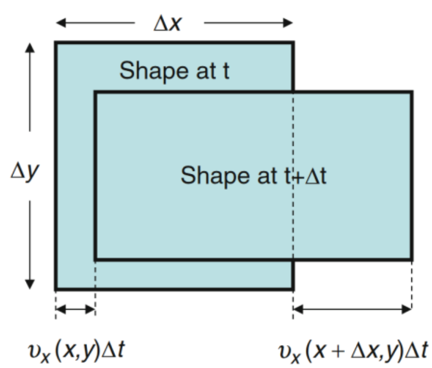


Fig. 24. 2-dimentional deformation of fluid when subjected to normal stress [1].

In this case, the normal strain is equal to the change in length of the object over the original length of the object:

$$\varepsilon_x = \frac{\Delta L}{L}$$

where ε_x is the normal strain in the x-direction, ΔL is the change in length, and L is the original length.

The change in length can be obtained by subtracting the original length at t by the length at $t + \Delta t$. The original length is simply Δx. The length at $t + \Delta t$ is the difference between the position of the right edge and the left:

$$length\ at\ t + \Delta t = (v_x(x + \Delta x, y)\Delta t + \Delta x) - v_x(x, y)\Delta t$$

So for the normal strain:

$$\varepsilon_x = \frac{\Delta L}{L} = \frac{(v_x(x + \Delta x, y)\Delta t + \Delta x - v_x(x, y)\Delta t) - \Delta x}{\Delta x} = \frac{(v_x(x + \Delta x, y)\Delta t - v_x(x, y)\Delta t)}{\Delta x}$$

Dividing by Δt and taking the limit:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{\varepsilon_x}{\Delta t} = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{\left(v_x(x + \Delta x, y) - v_x(x, y)\right)}{\Delta x}$$

$$\frac{\delta \varepsilon_x}{\delta t} = \frac{\delta v_x}{\delta x}$$

For the average rate of deformation, there is nothing to average over, so it is simply equal to the normal stress:

$$D_{xx} = \frac{\delta v_x}{\delta x}$$

If we repeat this in the other directions, it gives similar results:

$$D_{yy} = \frac{\delta v_y}{\delta y}$$

$$D_{zz} = \frac{\delta v_z}{\delta z}$$

Notice that, in general, the average rate of deformation is:

$$D_{ij} = D_{ji} = \frac{1}{2}\left(\frac{\delta v_i}{\delta j} + \frac{\delta v_j}{\delta i}\right)$$

Now, we need to relate the average rates of deformation to the shear stresses. In general:

$$\tau_{ij} = f\left(D_{ij}\right)$$

That function depends on the fluid we are studying. However, if **we assume an isotropic, incompressible fluid**, the equation is:

$$\tau_{ij} = -2\eta D_{ij} = -\eta\left(\frac{\delta v_i}{\delta j} + \frac{\delta v_j}{\delta i}\right)$$

where η is the apparent viscosity, which is a function of τ_{ij} itself.

If we **assume the fluid is Newtonian**:

$$\tau_{ij} = -\mu\left(\frac{\delta v_i}{\delta j} + \frac{\delta v_j}{\delta i}\right)$$

where μ is the viscosity, which is constant.

We can replace this into our original equations of motion. Here, we'll do it to the relevant part (the part containing shear stresses) of the x-momentum equation as an example.

$$-\frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} = \frac{\delta}{\delta x}\left(\mu\left(\frac{\delta v_x}{\delta x} + \frac{\delta v_x}{\delta x}\right)\right) + \frac{\delta}{\delta y}\left(\mu\left(\frac{\delta v_x}{\delta y} + \frac{\delta v_y}{\delta x}\right)\right) + \frac{\delta}{\delta z}\left(\mu\left(\frac{\delta v_z}{\delta x} + \frac{\delta v_x}{\delta z}\right)\right)$$

$$-\frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} = \mu\left(\frac{\delta^2 v_x}{\delta x^2} + \frac{\delta^2 v_x}{\delta x^2} + \frac{\delta^2 v_x}{\delta y^2} + \frac{\delta^2 v_y}{\delta x \delta y} + \frac{\delta^2 v_z}{\delta x \delta z} + \frac{\delta^2 v_x}{\delta z^2}\right)$$

$$-\frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} = \mu\left(\frac{\delta^2 v_x}{\delta x^2} + \frac{\delta^2 v_x}{\delta y^2} + \frac{\delta^2 v_x}{\delta z^2}\right) + \mu\left(\frac{\delta}{\delta x}\left(\frac{\delta v_x}{\delta x} + \frac{\delta v_y}{\delta y} + \frac{\delta v_z}{\delta z}\right)\right)$$

This can be further simplified using the continuity equation:

$$\frac{\delta \rho}{\delta t} + \frac{\delta}{\delta x}(\rho v_x) + \frac{\delta}{\delta y}(\rho v_y) + \frac{\delta}{\delta z}(\rho v_z) = 0$$

Since we are assuming the fluid is incompressible, density does not vary with time or position, so it can be taken out of the derivatives:

$$\frac{\delta v_x}{\delta x} + \frac{\delta v_y}{\delta y} + \frac{\delta v_z}{\delta z} = 0$$

So if we come back to our simplification:

$$-\frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} = \mu\left(\frac{\delta^2 v_x}{\delta x^2} + \frac{\delta^2 v_x}{\delta y^2} + \frac{\delta^2 v_x}{\delta z^2}\right) + \mu\left(\frac{\delta}{\delta x}(0)\right)$$

$$-\frac{\delta \tau_{xx}}{\delta x} - \frac{\delta \tau_{yx}}{\delta y} - \frac{\delta \tau_{zx}}{\delta z} = \mu\left(\frac{\delta^2 v_x}{\delta x^2} + \frac{\delta^2 v_x}{\delta y^2} + \frac{\delta^2 v_x}{\delta z^2}\right)$$

This can be repeated for the y- and z-direction momentum. All in all, the continuity equation and Navier Stokes equations for an **incompressible, isotropic, Newtonian** fluid are:

$$\frac{\delta v_x}{\delta x}+\frac{\delta v_y}{\delta y}+\frac{\delta v_z}{\delta z}=0$$

$$\rho\frac{\delta v_x}{\delta t}+\rho\left(v_x\frac{\delta v_x}{\delta x}+v_y\frac{\delta v_x}{\delta y}+v_z\frac{\delta v_x}{\delta z}\right)=\mu\left(\frac{\delta^2 v_x}{\delta x^2}+\frac{\delta^2 v_x}{\delta y^2}+\frac{\delta^2 v_x}{\delta z^2}\right)-\frac{\delta P}{\delta x}+\rho g_x$$

$$\rho\frac{\delta v_y}{\delta t}+\rho\left(v_x\frac{\delta v_y}{\delta x}+v_y\frac{\delta v_y}{\delta y}+v_z\frac{\delta v_y}{\delta z}\right)=\mu\left(\frac{\delta^2 v_y}{\delta x^2}+\frac{\delta^2 v_y}{\delta y^2}+\frac{\delta^2 v_y}{\delta z^2}\right)-\frac{\delta P}{\delta y}+\rho g_y$$

$$\rho\frac{\delta v_z}{\delta t}+\rho\left(v_x\frac{\delta v_z}{\delta x}+v_y\frac{\delta v_z}{\delta y}+v_z\frac{\delta v_z}{\delta z}\right)=\mu\left(\frac{\delta^2 v_z}{\delta x^2}+\frac{\delta^2 v_z}{\delta y^2}+\frac{\delta^2 v_z}{\delta z^2}\right)-\frac{\delta P}{\delta z}+\rho g_z$$

These can also be written as:

$$\nabla\cdot\vec{v}=0,\qquad \rho\frac{\delta\vec{v}}{\delta t}+\rho(\vec{v}\cdot\nabla\vec{v})=\mu\nabla^2\vec{v}-\nabla P+\rho\vec{g}$$

Note that the $\rho\left(v_x\frac{\delta v_x}{\delta x}+v_y\frac{\delta v_x}{\delta y}+v_z\frac{\delta v_x}{\delta z}\right)$ terms represent convective momentum transport or inertial effects and the $\mu\left(\frac{\delta^2 v_x}{\delta x^2}+\frac{\delta^2 v_x}{\delta y^2}+\frac{\delta^2 v_x}{\delta z^2}\right)$ terms represent diffusive momentum transfer or viscous effects.

We can also write Navier-Stokes as:

$$\rho\frac{D\vec{v}}{Dt}=\mu\nabla^2\vec{v}-\nabla P+\rho\vec{g}$$

See the section on the substantial derivative for more details.

Other Coordinate Systems

Cylindrical Coordinates

Continuity equation (general):

$$\frac{\delta\rho}{\delta t}+\frac{1}{r}\frac{\delta}{\delta r}(r\rho v_r)+\frac{1}{r}\frac{\delta}{\delta\theta}(\rho v_\theta)+\frac{\delta}{\delta z}(\rho v_z)=0$$

Continuity equation (for an incompressible fluid):

$$\frac{1}{r}\frac{\delta}{\delta r}(r v_r)+\frac{1}{r}\frac{\delta}{\delta\theta}(v_\theta)+\frac{\delta}{\delta z}(v_z)=0$$

Navier-Stokes equations:

$$\rho\frac{\delta v_r}{\delta t}+\rho\left(v_r\frac{\delta v_r}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_r}{\delta\theta}-\frac{v_\theta^2}{r}+v_z\frac{\delta v_r}{\delta z}\right)$$
$$=\mu\left(\frac{\delta}{\delta r}\left(\frac{1}{r}\frac{\delta}{\delta r}(rv_r)\right)+\frac{1}{r^2}\frac{\delta^2 v_r}{\delta\theta^2}-\frac{2}{r^2}\frac{\delta v_\theta}{\delta\theta}+\frac{\delta^2 v_r}{\delta z^2}\right)-\frac{\delta P}{\delta r}+\rho g_r$$

$$\rho\frac{\delta v_\theta}{\delta t}+\rho\left(v_r\frac{\delta v_\theta}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_\theta}{\delta\theta}-\frac{v_r v_\theta}{r}+v_z\frac{\delta v_\theta}{\delta z}\right)$$
$$=\mu\left(\frac{\delta}{\delta r}\left(\frac{1}{r}\frac{\delta}{\delta r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\delta^2 v_\theta}{\delta\theta^2}+\frac{2}{r^2}\frac{\delta v_r}{\delta\theta}+\frac{\delta^2 v_\theta}{\delta z^2}\right)-\frac{1}{r}\frac{\delta P}{\delta\theta}+\rho g_\theta$$

$$\rho\frac{\delta v_z}{\delta t}+\rho\left(v_r\frac{\delta v_z}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_z}{\delta\theta}+v_z\frac{\delta v_z}{\delta z}\right)=\mu\left(\frac{1}{r}\frac{\delta}{\delta r}\left(r\frac{\delta v_z}{\delta r}\right)+\frac{1}{r^2}\frac{\delta^2 v_z}{\delta\theta^2}+\frac{\delta^2 v_z}{\delta z^2}\right)-\frac{\delta P}{\delta z}+\rho g_z$$

Spherical coordinates

Continuity equation (general):

$$\frac{\delta\rho}{\delta t}+\frac{1}{r^2}\frac{\delta}{\delta r}(r^2\rho v_r)+\frac{1}{r\sin\theta}\frac{\delta}{\delta\theta}(\rho v_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\delta}{\delta\phi}(\rho v_\phi)=0$$

Continuity equation (for an incompressible fluid):

$$\frac{1}{r^2}\frac{\delta}{\delta r}(r^2 v_r)+\frac{1}{r\sin\theta}\frac{\delta}{\delta\theta}(v_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\delta}{\delta\phi}(v_\phi)=0$$

Navier-Stokes equations:

$$\rho\frac{\delta v_r}{\delta t}+\rho\left(v_r\frac{\delta v_r}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_r}{\delta\theta}-\frac{v_\phi^2+v_\theta^2}{r}+\frac{v_\phi}{r\sin\theta}\frac{\delta v_r}{\delta\phi}\right)$$
$$=\mu\left(\nabla^2 v_r-\frac{2}{r^2}\left(v_r+\frac{\delta v_\theta}{\delta\theta}+v_\theta\cot\theta+\frac{1}{\sin\theta}\frac{\delta v_\phi}{d\phi}\right)\right)-\frac{\delta P}{\delta r}+\rho g_r$$

$$\rho\frac{\delta v_\theta}{\delta t}+\rho\left(v_r\frac{\delta v_\theta}{\delta r}+\frac{v_\theta}{r}\left(v_r+\frac{\delta v_\theta}{\delta\theta}\right)+\frac{v_\phi}{r\sin\theta}\left(\frac{\delta v_\theta}{\delta\phi}-v_\phi\cos\theta\right)\right)$$
$$=\mu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\delta v_r}{\delta\theta}-\frac{1}{r^2\sin^2\theta}(v_\theta+2\cos\theta\frac{\delta v_\phi}{\delta\phi})\right)-\frac{1}{r}\frac{\delta P}{\delta\theta}+\rho g_\theta$$

$$\rho\frac{\delta v_\phi}{\delta t}+\rho\left(v_r\frac{\delta v_\phi}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_\phi}{\delta\theta}+\frac{v_\phi}{r\sin\theta}\left(\frac{\delta v_\phi}{\delta\phi}+v_r\sin\theta+v_\theta\cos\theta\right)\right)$$
$$=\mu\left(\nabla^2 v_\phi-\frac{1}{r^2\sin^2\theta}\left(v_\phi-2\sin\theta\frac{\delta v_r}{\delta\phi}-2\cos\theta\frac{\delta v_\theta}{\delta\phi}\right)\right)-\frac{1}{r\sin\theta}\frac{\delta P}{\delta z}+\rho g_\phi$$

Using Navier-Stokes

The Navier-Stokes equation does look scary at first, and it should. However, in this class, the challenge isn't to solve the equation, but to simplify it until it can be solved easily. Here is the general method to solving a general problem with Navier-Stokes:

1. Write down assumptions
2. Choose coordinate system if one isn't given
3. Use the assumptions to remove terms and simplify equations.
4. If necessary, scale the problem to simplify it further
5. Identify boundary conditions (need 1 for each variable and each derivative of a variable)
6. Solve equations

Examples of assumptions

- **Fluid properties:**
  - Incompressible fluid: can remove density term from continuity equation
  - Newtonian fluid: can use Navier-Stokes
- **Time-dependence:**
  - Steady state: can remove any term with $\frac{\delta}{\delta t}$.
- **Velocity:**
  - Only velocity in i-direction: can remove any term containing velocity in any other direction
  - fully developed flow: can sometimes remove velocities in other directions
  - No i-direction velocity: can remove any term containing v_i
- **Directional dependence:**
  - Property X doesn't vary in i-direction (for example, pressure doesn't change with z): remove any term with $\frac{\delta X}{\delta i}$
- **Gravity:**
  - Ignore gravity: remove gravity terms
  - You will have to change the gravity terms to fit your coordinate system regardless. For example, if your coordinate system is the "normal" cartesian coordinates, where the y-axis is vertical, then only g_y would be relevant, and you could remove g_x and g_z.
- **Scaling:**
  - Some side A is much longer than the other side B: $\frac{B}{A} \sim 0$. Usually, derivatives in the directions of the much longer side are negligible.
  - The Reynolds number is small or large.

Scaling

We've already looked at scaling previously. However, we only did it for 1D flow. If you are considering other dimensions, the process is very much the same, but you will have to find characteristic lengths for y and z, and characteristic velocities for v_y and v_z. Typically, you can simply use the height and width of the problem for y and z, and y- and z-direction average velocities for v_y and v_z. Typically, after substituting your nondimensional numbers in, you should look for the Reynolds number and ratios of characteristic lengths (like $\frac{height}{width}$, for example).

Stream Function

In 2D, for an incompressible fluid, we can define a stream function $\Psi(x, y)$ that satisfies the continuity equation:

$$v_x = \frac{\delta\Psi}{\delta y}, \qquad v_y = -\frac{\delta\Psi}{\delta x}$$

We can verify that this satisfies the continuity equation for incompressible fluids:

$$\frac{\delta v_x}{\delta x} + \frac{\delta v_y}{\delta y} = 0$$

$$\frac{\delta}{\delta x}\left(\frac{\delta\Psi}{\delta y}\right) + \frac{\delta}{\delta y}\left(-\frac{\delta\Psi}{\delta x}\right) = 0$$

$$\frac{\delta^2\Psi}{\delta x\delta y} - \frac{\delta^2\Psi}{\delta y\delta x} = 0$$

Streamlines

Let's start by taking a look at the differential of the stream function:

$$d\Psi = \frac{\delta\Psi}{\delta x}dx + \frac{\delta\Psi}{\delta y}dy = -v_y dx + v_x dy$$

If we take a line where the stream function is constant, such that $\Psi(x, y) = C$, then along this line the differential would be 0, since the value of Ψ would not change:

$$d\Psi = 0 = -v_y dx + v_x dy$$

If we rearrange:

$$\frac{dy}{dx}|_{\Psi=C} = \frac{v_y}{v_x}$$

This means that, at all points along this streamline, **the streamline is always tangent to the velocity field.** Fluid moves along the streamlines, and never crosses through it. This can be useful to visualize the path the fluid takes.

Reynolds Transport Theorem

We derived the equations of motion using a general shell balance approach, but there are other ways to derive the equation. Here we will explain the Reynolds transport theorem and apply it to derive the equations of motion again.

First, let's set up the scene. Let Φ be some extensive property, such as mass or energy, and ϕ be the corresponding intensive property, which is just Φ per unit volume, such as density or specific energy. Now let's take an arbitrary control volume, which is just an open system denoted CV, which has a bounding surface CS. Initially, within this control volume, there is a material volume, which is just a closed system denoted MV. The material volume contains a certain amount of Φ which, because it is a closed system, cannot flow in or out of the material volume. Initially, these two systems overlap, such that all of the property Φ in the material volume is also contained in the control volume. After a short time dt, however, the material volume moves compared to the control volume.

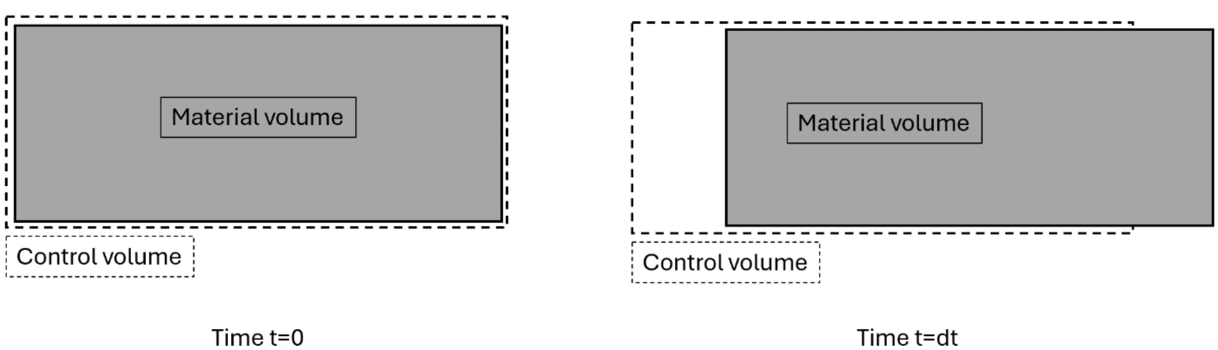
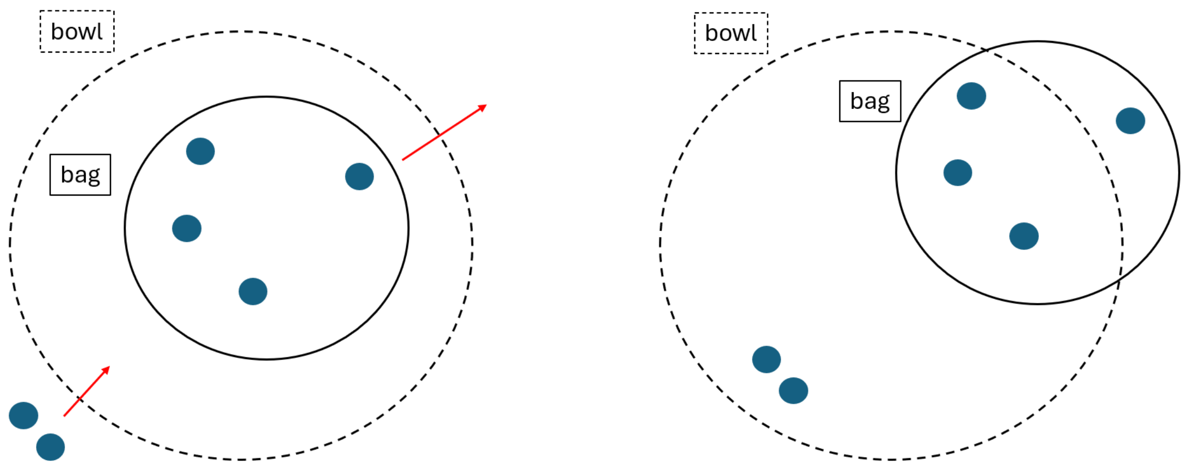


Fig. 25. Diagram of the movement of the material volume relative to the control volume at times 0 and dt.

We want to relate the rate of change of Φ in the material volume to that of the control volume. For now, let's start with a simple example with marbles. We initially have 4 marbles inside a closed bag within a bowl. Here, the marbles are the extensive property Φ, the bag is the material volume MV, and the bowl is the control volume CV. Since the material volume is a closed system, no marbles can enter or leave the bag. Let's say after some time dt, we have moved the bag a little such that one of the marbles within the bag has left the bowl, and have added two marbles to the bowl.



t=0
Marbles in bag: 4
Marbles in bowl: 4

t=dt
Marbles in bag: 4
Marbles in bowl: 5

Fig. 26. Illustration of the change in marbles inside the control volume (bowl) and material volume (bag).

We want to relate the change of marbles in the bag, which is 0, to the change of marbles in bowl, which is 1. We can do that using the net outflow of marbles out of the bowl, or the net amount of marbles which leave the bowl, which in this case is -1. In this case, then, we can say that:

$$\begin{Bmatrix} change\ of\ marbles \\ in\ the\ bag \end{Bmatrix} = \begin{Bmatrix} change\ of\ marbles \\ in\ the\ bowl \end{Bmatrix} + \begin{Bmatrix} net\ amount\ of\ marbles \\ which\ left\ the\ bowl \end{Bmatrix}$$

$$0 = 1 + (-1)$$

Now, if we generalize this, we can obtain the Reynolds transport theorem:

$$\begin{Bmatrix} rate\ of\ change\ of\ \Phi \\ in\ material\ volume \end{Bmatrix} = \begin{Bmatrix} rate\ of\ change\ of\ \Phi \\ in\ control\ volume \end{Bmatrix} + \begin{Bmatrix} net\ rate\ of\ outflow\ of\ \Phi \\ across\ boundary\ surface \end{Bmatrix}$$

which we can express as:

$$\frac{d}{dt}\int_{MV} \phi dV = \frac{d}{dt}\int_{CV} \phi dV + \int_{CS} \phi \vec{v} \cdot \vec{n} dA$$

where $\vec{v}$ is the velocity of the fluid relative to the boundary surface, and $\vec{n}$ is the unit normal vector pointing outwards.

Applying the Reynolds Transport Theorem

We can now use the Reynolds transport theorem to prove our equations of motion again. To do this, we apply the theorem to mass and momentum.

First, let's take Φ to be mass m and ϕ to be density ρ. Since mass is neither created nor destroyed, there is no way for the mass in the closed material volume to ever change, so we already know that:

$$\begin{Bmatrix} rate\ of\ change\ of\ \Phi \\ in\ material\ volume \end{Bmatrix} = 0$$

which leaves us with:

$$0 = \frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \vec{v} \cdot \vec{n} dA$$

Now, we can simplify this. First, we can use Leibniz rule to take the derivative inside the integral:

$$\frac{d}{dt}\int_{CV} \rho dV = \int_{CV} \frac{\delta\rho}{\delta t} dV$$

Second, we can use the divergence theorem (think back to MATH 264) to change the surface integral into a volume integral. As a reminder, the divergence or Gauss's theorem is:

$$\int_{V} \nabla \cdot \vec{F} dV = \int_{S} \vec{F} \cdot \hat{n} dS$$

In our case:

$$\int_{CS} \rho \vec{v} \cdot \vec{n} dA = \int_{CV} \nabla \cdot (\rho \vec{v}) dV$$

which leaves us with:

$$0 = \int_{CV} \frac{\delta\rho}{\delta t} dV + \int_{CV} \nabla \cdot (\rho \vec{v}) dV = \int_{CV} \left(\frac{\delta\rho}{\delta t} + \nabla \cdot (\rho \vec{v}) \right) dV$$

Now we can just get rid of the integral to come to:

$$\frac{\delta\rho}{\delta t} + \nabla \cdot (\rho \vec{v}) = 0$$

This is our continuity equation.

Now, let's do the same thing for momentum, so ϕ is momentum density $\rho\vec{v}$. Momentum can be created by shear and gravitational forces, so in our case:

$$\begin{Bmatrix} rate\ of\ change\ of\ \Phi \\ in\ material\ volume \end{Bmatrix} = \int_{CV} \rho\vec{g}dV + \int_{CS} \underline{\sigma} \cdot \vec{n}dA$$

where $\vec{g}$ is the acceleration due to gravity and $\underline{\sigma}$ is the stress tensor.

Putting this into the Reynolds transport theorem:

$$\int_{CV} \rho\vec{g}dV + \int_{CS} \underline{\sigma} \cdot \vec{n}dA = \frac{d}{dt}\int_{CV} \rho\vec{v}dV + \int_{CS} \rho\vec{v}(\vec{v}\cdot\vec{n})dA$$

We once again use Leibniz rule (and the product rule):

$$\frac{d}{dt}\int_{CV} \rho\vec{v}dV = \int_{CV} \frac{\delta}{\delta t}(\rho\vec{v})dV = \int_{CV} \left(\frac{\delta\rho}{\delta t}\vec{v} + \rho\frac{\delta\vec{v}}{\delta t}\right)dV$$

Using the divergence theorem:

$$\int_{CS} \underline{\sigma} \cdot \vec{n}dA = \int_{CV} \nabla \cdot \underline{\sigma}dV$$

$$\int_{CS} \rho\vec{v}(\vec{v}\cdot\vec{n})dA = \int_{CV} \nabla\cdot(\rho\vec{v}\vec{v})dV = \int_{CV} (\vec{v}\nabla\cdot(\rho\vec{v}) + (\rho\vec{v})\cdot\nabla\vec{v})dV$$

Together, it gives:

$$\int_{CV} \rho\vec{g}dV + \int_{CV} \nabla\cdot\underline{\sigma}dV = \int_{CV} \left(\frac{\delta\rho}{\delta t}\vec{v} + \rho\frac{\delta\vec{v}}{\delta t}\right)dV + \int_{CV} (\vec{v}\nabla\cdot(\rho\vec{v}) + (\rho\vec{v})\cdot\nabla\vec{v})dV$$

We can get rid of the integrals:

$$\frac{\delta\rho}{\delta t}\vec{v} + \rho\frac{\delta\vec{v}}{\delta t} + \vec{v}\nabla\cdot(\rho\vec{v}) + (\rho\vec{v})\cdot\nabla\vec{v} = \rho\vec{g} + \nabla\cdot\underline{\sigma}$$

$$\vec{v}\left(\frac{\delta\rho}{\delta t} + \nabla\cdot(\rho\vec{v})\right) + \rho\left(\frac{\delta\vec{v}}{\delta t} + \vec{v}\cdot\nabla\vec{v}\right) = \rho\vec{g} + \nabla\cdot\underline{\sigma}$$

We've already proven that $\frac{\delta\rho}{\delta t} + \nabla\cdot(\rho\vec{v}) = 0$:

$$\rho\left(\frac{\delta\vec{v}}{\delta t} + \vec{v}\cdot\nabla\vec{v}\right) = \rho\vec{g} + \nabla\cdot\underline{\sigma}$$

We can express $\frac{\delta\vec{v}}{\delta t} + \vec{v}\cdot\nabla\vec{v}$ using the material derivative $\frac{D\vec{v}}{Dt}$. In addition, for a viscous fluid, the stress tensor $\underline{\sigma}$ can be expressed as:

$$\underline{\sigma} = -P\underline{I} + \underline{\tau}$$

where P is pressure, $\underline{I}$ is the identity matrix, and $\underline{\tau}$ is the shear stress from viscosity. Thus, the stress tensor term becomes:

$$\nabla \cdot \underline{\sigma} = \nabla \cdot -P\underline{I} + \nabla \cdot \underline{\tau} = -\nabla P + \nabla \cdot \underline{\tau}$$

This leaves us with:

$$\frac{D\vec{v}}{Dt} = \rho\vec{g} - \nabla P + \nabla \cdot \underline{\tau}$$

which is a way to write the Cauchy momentum equation. If we use the incompressible Newtonian fluid assumption as we have done above, we will come to the Navier-Stokes equation.

Heat Transfer

Introduction

Heat transfer occurs in three mechanisms. Two of these are familiar: “diffusion” in the form of heat conduction, and convective heat transfer from the bulk movement of a fluid carrying some heat. In addition, heat can be transferred through radiation, or electromagnetic waves. To start, we’ll go through these one by one.

Heat conduction

Heat conduction is caused by rapidly vibrating molecules which can interact with neighboring molecules, causing them to vibrate. In the case of fluids, there is also the random movement of molecules which creates diffusion of rapidly and slowly vibrating molecules. As we’ve seen previously, for diffusive transport, the flux is proportional to the gradient of some potential. Here, the heat flux is proportional to the temperature gradient, with the thermal conductivity as the proportionality constant, giving us **Fourier’s law of conduction**.

$$q_x = \frac{\dot{Q}_x}{A} = -k\frac{dT}{dx}$$

where q_x is the heat flux in the x-direction, $\dot{Q}_x$ is the heat flow rate in the x-direction, A is the area (heat flow is normal to this area), k is thermal conductivity, and T is temperature.

Example: temperature gradient in a hollow cylinder

The temperature gradient is easy enough to obtain in rectangular coordinates by applying Fourier’s law of conduction, but might be a little trickier in cylindrical coordinates because the area changes as a function of the radius. Because of this, it’s better to work with the heat flow rate than the heat flux. Let’s use a simple example with some hollow cylinder of length L, inner radius r_i with inner temperature T_i, and outer radius r_o with outer temperature T_o. We wish to know the temperature gradient in the r-direction if we are at steady state.

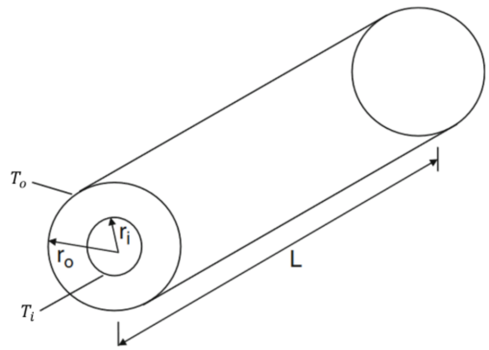


Fig. 27. Diagram of the hollow cylinder [1].

Let's start by using Fourier's law of conduction, but using heat flow rate rather than heat flux:

$$\dot{Q}_r = -kA\frac{dT}{dr}$$

where the lateral area of the cylinder A is a function of r:

$$A = 2\pi rL$$

Now, because we are at steady state, the heat flow rate should be constant. Let's take a thin cylindrical "slice" and think about the heat that goes in and out of this "slice". We should find that, if the heat going into the slice does not equal the heat coming out of it, heat would accumulate and temperature would change. This wouldn't work in our steady state case. Then the heat going into and out of any arbitrary "slice" have to be equal, so heat flow rate must be constant over the radius of the cylinder.

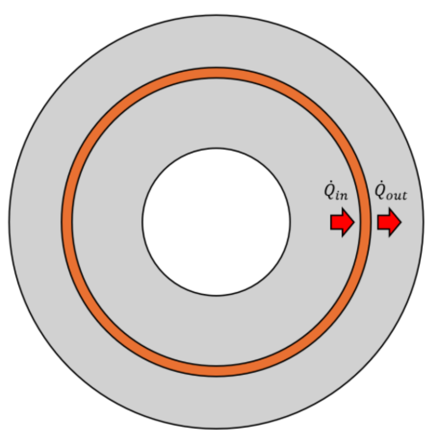


Fig. 28. Heat flow rate into and out of thin slice of the hollow cylinder. At steady state, these must be equal.

From there, it's simply a matter of integrating and using our boundary conditions. First, integrating over r:

$$\frac{dT}{dr} = -\frac{\dot{Q}_r}{kA} = -\frac{\dot{Q}_r}{k(2\pi L)}\frac{1}{r}$$

$$T = \int -\frac{\dot{Q}_r}{k(2\pi L)}\frac{1}{r}dr = -\frac{\dot{Q}_r}{k(2\pi L)}\ln(r) + C_1$$

We don't know what the value of $\dot{Q}_r$ is, but we can obtain it using our boundary conditions:

$$T(r = r_i) = T_i$$

$$T(r = r_o) = T_o$$

$$T_o - T_i = -\frac{\dot{Q}_r}{k(2\pi L)}\ln(r_o) + \frac{\dot{Q}_r}{k(2\pi L)}\ln(r_i)$$

$$\dot{Q}_r = \frac{2\pi kL(T_o - T_i)}{\ln\left(\frac{r_i}{r_o}\right)}$$

which leaves us with:

$$T = -\frac{(T_o - T_i)}{\ln\left(\frac{r_i}{r_o}\right)}\ln(r) + C_1$$

We can find C_1 using either of our boundary conditions and come to:

$$\frac{T(r) - T_i}{T_o - T_i} = \frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)}$$

Temperature gradient in a slab

The process to find the temperature gradient of a slab (in cartesian coordinates) is similar, but area is independent of x, so we can work using the heat flux rather than heat flow rate. We wish to know the steady state temperature gradient of a slab of length L if the left side is maintained at T_o and the right side is maintained at T_L.

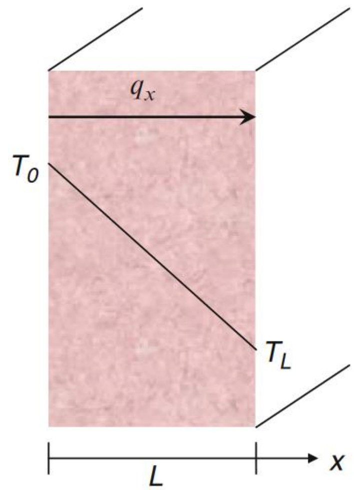


Fig. 29. Diagram of a slab and temperature distribution within a slab at steady state [1].

In this case, the temperature gradient will be:

$$T(x) = \frac{T_L - T_o}{L} x + T_o$$

Electrical circuits analogy

Just as in the fluid mechanics case, we can use an electrical circuit analogy. In this case, electrical current will be analogous to heat flow rate, electrical potential difference will be analogous to temperature difference, and electrical resistance will be analogous to thermal resistance to conduction R_T. Then, Ohm's law $\left(I = \frac{V}{R}\right)$ becomes:

$$\dot{Q} = \frac{\Delta T}{R_{T,cond}}$$

The expression for thermal resistance depends on the geometry of the object. In a slab:

$$\dot{Q} = \left(\frac{Ak}{L}\right)\Delta T = \frac{\Delta T}{R_{T,cond}}, \qquad R_{T,cond} = \frac{L}{Ak}$$

where A is area.

In a hollow cylinder:

$$\dot{Q} = \left(\frac{2\pi kL}{\ln\left(\frac{r_o}{r_i}\right)}\right)\Delta T = \frac{\Delta T}{R_{T,cond}}, \qquad R_{T,cond} = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi kL}$$

We can combine these resistors into circuits. Let's take the example of a wall made of three different materials as pictured below, with three different thermal conductivities. One side is maintained at a high temperature T_H and the other at a low temperature T_L. If we wish to know the heat flow rate $\dot{Q}$, we can make an equivalent circuit as pictured below in **Fig. 30**.

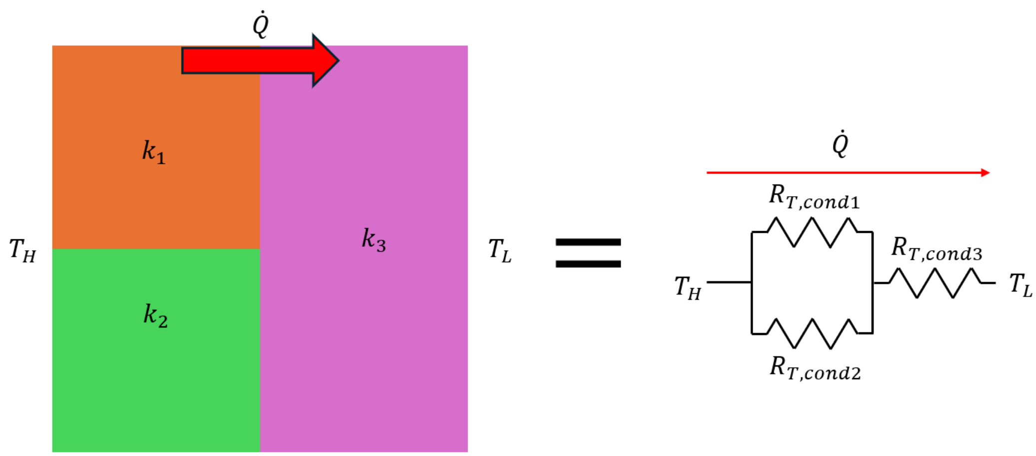


Fig. 30. Diagram of equivalent electrical circuit to a slab composed of three different materials with different electrical conductivities k_1, k_2, k_3.

Then, we can simply calculate the heat flow rate as:

$$\dot{Q} = \frac{\Delta T}{R_{T,total}} = \frac{\Delta T}{\dfrac{1}{\dfrac{1}{R_{T,cond1}} + \dfrac{1}{R_{T,cond2}}} + R_{T.cond3}}$$

Heat convection

When a fluid moves across a surface, the rate of heat transfer can be greatly enhanced. Think about wind chill: even if the outside air is at the same temperature, it feels much cooler when it's windy. This effect is due to heat convection.

We can quantify heat transfer at the surface due to heat convection using Newton's law of cooling:

$$q = \frac{\dot{Q}}{S} = h(T_s - T_\infty)$$

where q is heat flux at the surface, $\dot{Q}$ is heat flow rate at the surface, S is the surface area, T_s is the temperature at the surface, T_∞ is the temperature of the fluid far away from the surface, and h is the convective heat transfer coefficient.

Then, to deal with heat convection problems, the key is finding this heat transfer coefficient, which is usually obtained empirically. Before doing that, however, let's go over boundary layers.

Boundary layers

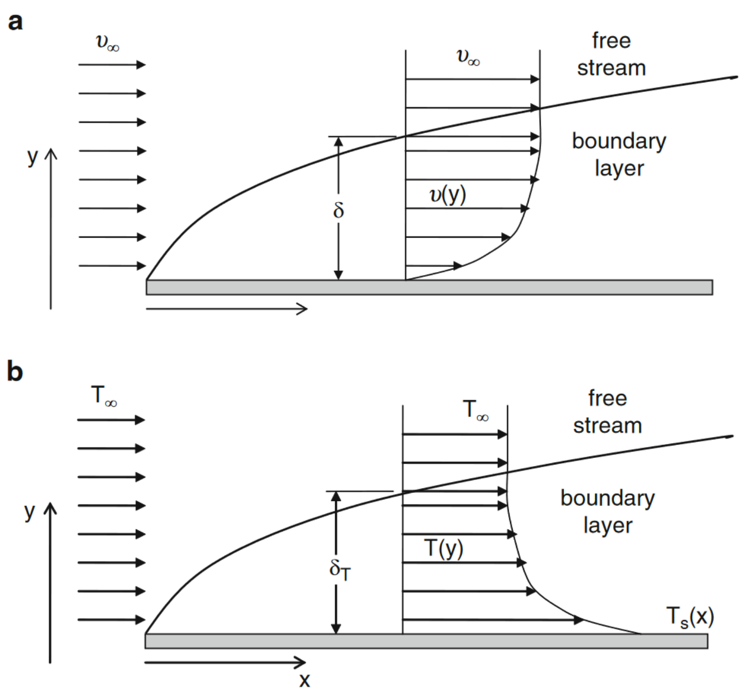


Fig. 31. Progression of the velocity and temperature boundary layer across a thin horizontal plate [1].

Let's imagine that some fluid is moving at velocity v_∞ with temperature T_∞ before encountering a flat plate which is unmoving at temperature T_s. Right at the edge, when the fluid first encounters the plate, the layer of fluid directly in contact with the plate is stopped due to the no slip condition. However, the layer of fluid directly above it isn't affected yet; it still moves at v_∞. As we move along the plate, momentum will have had the opportunity to diffuse, so the fluid layers close the surface will move slower than those high above it. Thus, we have a free stream high above the plate where fluid moves at v_∞, and a boundary layer close to the plate where fluid gradually slows down until being immobile at the plate surface. As we move further along the plate, momentum will diffuse higher and higher, and the boundary layer will expand. We define the edge of this boundary layer as the height where fluid velocity is at 99% of the free stream velocity v_∞. The height of the boundary layer is related to the Reynolds number:

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}}$$

where δ is the height of the boundary layer, L is the length along the plate, and Re_L is the Reynolds number.

So far, we have been talking about the velocity (or momentum) boundary layer, but a similar thing happens for temperature: at the edge, the fluid layer in contact with the surface will be at T_s, and as we move along the plate, heat will have had time to diffuse upwards, creating a thermal boundary layer. We define the edge of this boundary layer as the height where the difference between fluid temperature and surface temperature T_s is at 99% of difference between free stream fluid temperature T_∞ and surface temperature T_s $\left(\frac{T_s - T}{T_s - T_\infty} = 0.99\right)$. The height of the thermal and velocity boundary layers don't have to be the same. We can compare them using the Prandtl number (Pr), which compares momentum diffusivity to thermal diffusivity.

$$Pr = \frac{c_p \mu}{k} = \frac{\nu}{\alpha}$$

where Pr is the Prandtl number, c_p is the specific heat capacity, μ is viscosity, k is thermal conductivity, ν is the dynamic viscosity or momentum diffusivity (defined as $\nu = \frac{\mu}{\rho}$, where ρ is density), and α is thermal diffusivity (defined as $\alpha = \frac{k}{\rho c_p}$).

Thus, when the Prandtl number is small, momentum diffuses slower than heat, so the momentum boundary layer will be smaller than the thermal boundary layer, and vice versa. The height of the thermal boundary layer, then, can be obtained as:

$$\frac{\delta_T}{L} \sim \frac{1}{\sqrt{Re_L}\sqrt{Pr}}$$

where δ_T is the height of the boundary layer.

Finding the heat transfer coefficient

The heat transfer coefficient depends on multiple factors:

1. **Free or forced convection**
   - In forced convection, fluid flow is driven by an external source, like a fan, pump, or wind.
   - In free convection, the heat itself creates fluid flow due to thermal expansion of the fluid. Consider air above a hot plate: as the hot plate heats air above it, the hot air will expand and rise, and cool air will come in from the side of the plate. This happens anywhere there is gravity.
2. **Geometry of the boundary layer region**, and whether **flow is external or internal**
   - In external flow, the boundary layer will keep expanding.
   - In internal flow, the boundary layer is constrained; eventually the boundary layers on each side will meet, and flow will be fully developed.
   - In free convection, the orientation of the surface relative to gravity is important.
3. **Laminar or turbulent flow** in the boundary layer domain
4. **Properties of the fluid and surface**

When talking about the heat transfer coefficient, we can either refer to the local or average heat transfer coefficient. The local heat transfer coefficient h_x depends on the position along the surface, whereas the average heat transfer coefficient $\bar{h}$ is averaged over the surface:

$$\bar{h} = \frac{1}{S}\int_S h_x dS$$

where h_x is the local heat transfer coefficient, $\bar{h}$ is the average heat transfer coefficient, and S refers to the surface area.

As stated previously, the heat transfer coefficient has to be obtained empirically. Because heat convection problems include many variables, we use the Buckingham Pi theorem to make experiments easier. After applying Buckingham Pi, the relation used to find the heat transfer coefficient is:

$$\frac{hL}{k_f} = f\left(\frac{c_{pf}\mu}{k_f}, \frac{\rho_f \langle v \rangle L}{\mu}, \frac{\rho_f^2 L^3 \beta g (T_s - T_\infty)}{\mu^2}, \frac{d}{L}, \frac{k_s}{k_f}\right)$$

$$Nu_L = f\left(Pr, Re_L, Gr_L, \frac{d}{L}, \frac{k_s}{k_f}\right)$$

where h is the heat transfer coefficient, L and d are characteristic lengths, k_f and k_s are heat transfer coefficients of the fluid and surface respectively, ρ_f is the density of the fluid, c_{pf} is the specific heat capacity of the fluid, μ is viscosity, $\langle v \rangle$ is the fluid velocity in forced convection, β is the coefficient of volume expansion, g is gravitational acceleration, and T_s and T_∞ are the temperature of the surface and fluid respectively.

There are a lot of dimensionless numbers present. We have already seen the Reynolds number Re_L and Prandtl number Pr. The new numbers are the Nusselt number Nu_L and the Grashof number Gr_L. The Nusselt number is the ratio of heat convection to heat conduction inside the fluid, while the Grashof number is the ratio of buoyant to viscous forces, which is useful when dealing with free convection.

However, in general, not all of these terms are important. For forced convection, the equation simplifies to:

$$Nu_L = f(Re_L, Pr)$$

For free convection, it simplifies to:

$$Nu_L = f(Gr, Pr)$$

Heat transfer coefficient equations

In general, the method to obtain the heat transfer coefficient is:

1. Determine the characteristics of the system:
    - Forced or free convection
    - Internal or external flow
    - Laminar or turbulent flow
    - Geometry and orientation
2. Use the empirically derived equation for this situation to find the Nusselt number
    - Can be given in the problem, found in this coursepack, in lecture notes, or in the literature.
3. Use the Nusselt number to find the heat transfer coefficient

Situation: steady state, forced convection, internal flow, circular cross-section

In this situation, we need to consider both thermal and hydrodynamic entrance lengths.

Table 4: Thermal and hydrodynamic entrance lengths.

Entrance length	Laminar flow	Turbulent flow
Hydrodynamic entrance length	$\frac{L_{ent,h}}{d} \approx 0.0567 Re_d$	$\frac{L_{ent,h}}{d} \approx 10$
Thermal entrance length	$\frac{L_{ent,t}}{d} \approx 0.0567 Re_d Pr$	$\frac{L_{ent,t}}{d} \approx 10$

where $L_{ent,h}$ is the hydrodynamic entrance length, $L_{ent,t}$ is the thermal entrance length, and d is the conduit diameter.

In addition, since there is no "free stream temperature" T_∞ in internal flow, we use the mixing cup temperature, which is the temperature we would obtain if we took a thin slice of the fluid at the cross-section and mixed it together:

$$T_m = \frac{\int_{A_c} (\rho c_p T) v dA_c}{\int_{A_c} (\rho c_p) v dA_c}$$

where T_m is mixing cup temperature, ρ is density, c_p is specific heat capacity, T is temperature, v is velocity, and A_c is the cross-sectional area.

In the case of an incompressible fluid with constant c_p, in a circular cross-section of radius r_o, the mixing cup temperature simplifies to:

$$T_m = \frac{2 \int_0^{r_o} vTrdr}{\langle v \rangle r_o^2}$$

where $\langle v \rangle$ is the average velocity.

Table 5: Nusselt number at steady state in forced convection internal flow with circular cross-section for multiple cases

Case	Equation	Notes
Fully developed laminar flow, constant heat flux	$Nu_d = 4.36$	Thermal conductivity evaluated at mixing cup temperature
Fully developed laminar flow, constant temperature	$Nu_d = 3.66$	Thermal conductivity evaluated at mixing cup temperature
Velocity and temperature profiles developing, laminar flow	$\overline{Nu_d} = 3.66 + \frac{0.0668\left(\frac{d}{L}\right) Re_d Pr}{1 + 0.04\left[\left(\frac{d}{L}\right) Re_d Pr\right]^{2/3}}$	
Turbulent flow	$Nu_d = 0.023 Re_d^{4/5} Pr^{1/3}$	Usually, entry length is very short and can be ignored

Situation: Forced external flow

Fluid properties are evaluated at the film temperature T_f, which is the average of the surface temperature T_s and the free stream temperature T_∞:

$$T_f = \frac{T_s + T_\infty}{2}$$

Table 6: Nusselt number in forced convection external flow for multiple cases

<table>
<tr><th>Case</th><th>Equation</th><th>Notes</th></tr>
<tr><td>Flat plate, laminar</td><td>$Nu_x = \frac{h_x x}{k} = 0.332 Re_x^{1/2} Pr^{1/2}$</td><td>$Pr \geq 0.6$</td></tr>
<tr><td>Flat plate, turbulent</td><td>$Nu_x = \frac{h_x x}{k} = 0.0296 Re_x^{4/5} Pr^{1/3}$</td><td>$Re_x \leq 10^8$
$0.6 \leq Pr \leq 60$</td></tr>
<tr><td>Average over flat plate, laminar</td><td>$\overline{Nu_L} = 0.664 Re_L^{0.5} Pr^{0.33}$</td><td>$Pr \geq 0.6$</td></tr>
<tr><td>Average over flat plate, both laminar and turbulent regions</td><td>$\overline{Nu_L} = (0.037 Re_L^{0.8} - 871) Pr^{0.33}$</td><td>$Re_x \leq 10^8$
$0.6 \leq Pr \leq 60$</td></tr>
<tr><td>Average over perpendicular cylinder</td><td>$\overline{Nu_d} = C Re_d^n Pr^{0.33}$
<table>
<tr><th>Re_d</th><th>C</th><th>n</th></tr>
<tr><td>0.4-4</td><td>0.989</td><td>0.330</td></tr>
<tr><td>4-40</td><td>0.911</td><td>0.385</td></tr>
<tr><td>40-4000</td><td>0.683</td><td>0.466</td></tr>
<tr><td>4000-40000</td><td>0.193</td><td>0.618</td></tr>
<tr><td>40000-400000</td><td>0.027</td><td>0.805</td></tr>
</table></td><td>$0.4 \leq Re_d \leq 4 \times 10^5$
$Pr \geq 0.7$
d is cylinder diameter</td></tr>
<tr><td>Average over a sphere</td><td>$\overline{Nu_d} = 2 + \left(0.4 Re_d^{0.5} + 0.06 Re_d^{0.67}\right) \mathrm{Pr}^{0.4} \left(\frac{\mu}{\mu_s}\right)^{0.25}$</td><td>$3.5 \leq Re_d \leq 7.6 \times 10^4$
$0.6 \leq Pr \leq 60$
Properties evaluated at free stream temperature T_∞, except for μ_s which is viscosity at the surface temperature</td></tr>
</table>

Situation: free convection

In free convection, the Rayleigh number Ra_L is often used, which is the ratio of free convection to conduction in the fluid:

$$Ra_L = \frac{g\beta(T_s - T_\infty)L^3}{\alpha\nu} = Gr_L Pr$$

where Ra_L is the Rayleigh number, g is acceleration due to gravity, β is the coefficient of volume expansion (1/T for ideal gases), T_s is the surface temperature, T_∞ is the free stream temperature, L is the characteristic length, α is the thermal diffusivity, ν is kinematic viscosity $\frac{\mu}{\rho}$, Gr_L is the Grashof number, and Pr is the Prandtl number.

Table 7: Nusselt number at steady state in free convection for multiple cases

Case	Equation	Notes
Vertical plate, laminar	$\overline{Nu_L} = 0.59Ra_L^{1/4}$	$10^4 \leq Ra \leq 10^9$
Vertical plate, turbulent	$\overline{Nu_L} = 0.1Ra_L^{1/3}$	$10^9 \leq Ra \leq 10^{13}$
Vertical plate, laminar and turbulent	$\overline{Nu_L} = \left\{0.825 + \frac{0.387Ra_L^{\frac{1}{6}}}{\left[1 + \left(\frac{0.492}{Pr}\right)^{\frac{9}{16}}\right]^{\frac{8}{27}}}\right\}^2$	
Horizontal plate, upper surface of a hot plate/ lower surface of a cold plate	$\overline{Nu_L} = 0.54Ra_L^{1/4}$ $\overline{Nu_L} = 0.15Ra_L^{1/3}$	$10^4 \leq Ra \leq 10^7$ $10^7 \leq Ra \leq 10^{11}$
Horizontal plate, lower surface of a hot plate/ upper surface of a cold plate	$\overline{Nu_L} = 0.27Ra_L^{1/4}$	$10^5 \leq Ra \leq 10^{10}$
External flow over a horizontal cylinder	$\overline{Nu_d} = \left\{0.6 + \frac{0.387Ra_L^{\frac{1}{6}}}{\left[1 + \left(\frac{0.559}{Pr}\right)^{\frac{9}{16}}\right]^{\frac{8}{27}}}\right\}^2$	Isothermal surface, diameter d $Ra \leq 10^{12}$

External flow over a sphere	$$\overline{Nu_d} = 2 + \frac{0.589 Ra_d^{1/4}}{\left[1 + \left(\frac{0.469}{Pr}\right)^{\frac{9}{16}}\right]^{\frac{4}{9}}}$$	Isothermal surface, diameter d $Ra \leq 10^{11}$ $Pr \leq 0.7$

Electrical circuit analogy

We can use an electrical circuit analogy for Newton's law of cooling. Remember that:

$$q = \frac{\dot{Q}}{S} = h(T_s - T_\infty)$$

which we can rearrange into:

$$\dot{Q} = hS(T_s - T_\infty) = \frac{\Delta T}{R_{T,conv}}, \qquad R_{T,conv} = \frac{1}{hS}$$

We can combine these into circuits. For example, let's take a heat exchanger, where there is water flowing on both sides of a plate, with one side having hot water and the other having cold water.

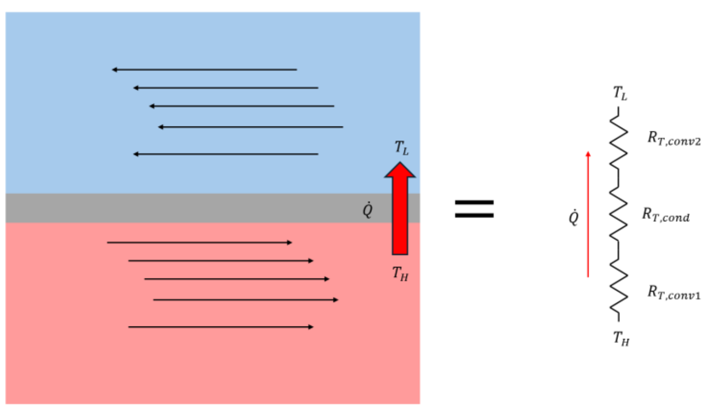


Fig. 32. Diagram of equivalent electrical circuit to heat transport across a heat exchanger.

In this case, heat is transferred through convection from the hot water to the inside of the plate, then conduction in the plate from one side to the other, and finally convection from the outside of the plate to the cold water. We can make an equivalent circuit of resistors in series, as pictured, making it easy to determine the total heat flow rate:

$$\dot{Q} = \frac{\Delta T}{R_{T,total}} = \frac{\Delta T}{R_{T,conv1} + R_{T,cond} + R_{T,conv2}}$$

Radiation

The third mechanism for heat exchange is thermal radiation. All surfaces above 0 Kelvin emit electromagnetic radiation, and all surfaces constantly receive thermal energy from incoming electromagnetic radiation. The rate of radiation heat transfer is governed by the surface temperature, the surface radiation properties, and the size, shape, and orientation of the object.

Black body radiation

A black body is a theoretical object which perfectly absorbs all light and reflects nothing, such that it appears black, and emits the maximum amount of radiation possible for a certain temperature. For black bodies, the radiant emittance, or the rate of radiant energy emitted per unit area (or energy flux) is given by the Stefan-Boltzmann law:

$$E_b = \sigma T^4$$

where E_b is radiant emittance of a black body, σ is the Stefan-Boltzmann constant of $5.67 \times 10^{-8} \frac{W}{m^2K^4}$, and T is the surface temperature.

Surface properties

Real surfaces aren't black bodies. They typically don't absorb all light and typically emit less radiation than a black body. We can define the emissivity as the ratio of radiant emittance of a real body to that of a black body.

$$\varepsilon = \frac{E}{E_b}$$

where ε is the emittance, E is the radiant emittance of a real body, and E_b is the radiant emittance of a black body.

Note that, in general, emittance depends both on temperature and wavelength. However, we will typically assume that we are working with gray bodies, which have properties independent of wavelength, so emissivity only depends on temperature.

Now let's turn to the case where light is incident upon an object. In this case light can be transmitted through the object, absorbed by the object, and/or reflected. Then, we can define three coefficients:

- Coefficient of **absorption**, α, is the ratio of radiant flux absorbed over the incident radiant flux

- Coefficient of **reflection**, ρ, is the ratio of radiant flux reflected over the incident radiant flux
- Coefficient of **transmission**, τ, is the ratio of radiant flux transmitted over the incident radiant flux

The addition of these should be 1.

$$\alpha + \rho + \tau = 1$$

According to Kirchoff's identity, for a gray surface, the emissivity and absorption coefficient are equal (hence why black bodies emit the maximum radiation they can; their absorptivity and emissivity are 1):

$$\varepsilon = \alpha$$

Geometry

If we take two bodies, not all of the radiation emitted from one body will impinge upon another. I am not getting the full concentrated power of the sun standing outside, but only the small part of it which reaches me. To determine the fraction of emitted radiation that hits another body, we have to take into account their sizes, shapes, orientation from each other, and relative distance. We can concentrate these factors into a shape factor, $F_{n \to m}$, which is defined as the fraction of radiant energy leaving object n that is incident upon object m.

Note that, in general:

$$F_{n \to m} \neq F_{m \to n}$$

These shape factors have three relations:

1. **Reciprocity relation:**
$$A_n F_{n \to m} = A_m F_{m \to n}$$
where A_i is the surface area of object i.
2. **Summation rule:** the sum of the shape factor from a body to the complete environment is 1, i.e. represents all the energy leaving the object.
$$\sum_{k=1}^{n} F_{m \to k} = 1$$
3. **Additive rule:** we can split a surface into components, in which case the shape factor incident upon the object will be equal to the sum of the shape factors incident upon the components. For an object n split into i components:

$$F_{m \to n} = \sum_{j=1}^{i} F_{m \to j}$$

Calculating these shape factors involves a lot of math, but thankfully others have done the work for us and put everything into simple tables and graphs. For example, here is the shape factor for two sheets at a right angle from each other. Other graphs can be found in the Roselli and Diller textbook, or in the literature.

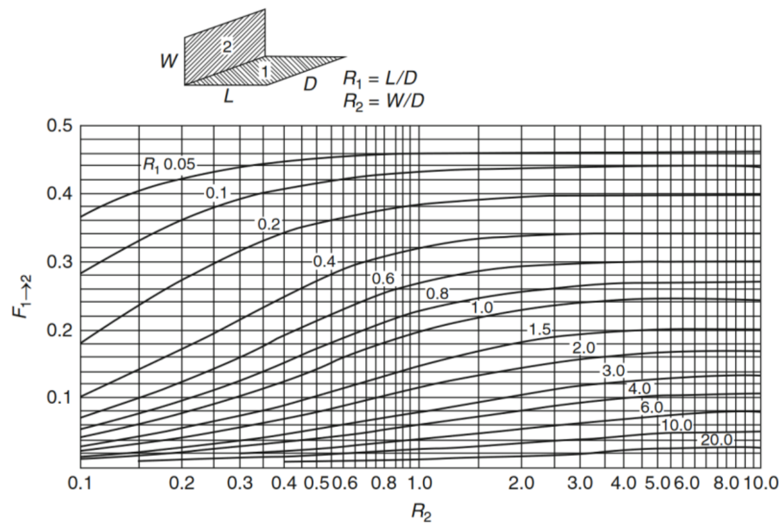


Fig. 33. Shape factor for radiation between aligned, adjacent, perpendicular rectangles as a function of their relative sizes and separations [1].

Putting it all together

First, let's consider a single surface, receiving an irradiation G, which is the rate at which radiant energy is hitting the surface per unit area. This object has a radiosity J, which is the rate at which radiant energy exits the object per unit area. Let's find the net exchange of energy, or net heat flux, on that surface, which is simply the difference between the irradiation and the radiosity. Let's assume we are at steady state, the surface does not transmit light, and is a gray body.

Let's clarify what happens to the surface. Some irradiation G enters the surface. Part of it is absorbed, while another part is reflected and leaves the object. At the same time, some radiant energy leaves the object due to thermal radiation. The radiosity is the sum of the energy generated from thermal radiation and the reflected irradiation:

$$J = \varepsilon E_b + \rho G$$

which we can rearrange into:

$$G = \frac{J - \varepsilon E_b}{\rho}$$

We can replace the reflectivity ρ. Remember that the sum of reflectivity, transmittivity, and absorptivity are 1. There is no light transmitted here, however, so:

$$\alpha + \rho = 1$$

$$\rho = 1 - \alpha$$

We can make use of Kirchoff's identity ($\alpha = \varepsilon$):

$$\rho = 1 - \varepsilon$$

which allows us to rewrite the irradiation equation into:

$$G = \frac{J - \varepsilon E_b}{1 - \varepsilon}$$

We know that the heat flux will be the difference between energy influx, or irradiation, and energy outflux, or radiosity:

$$q = \frac{\dot{Q}}{A} = J - G$$

where q is heat flux, $\dot{Q}$ is heat flow rate, and A is surface area.

We can simply substitute the radiosity we found earlier and rearrange:

$$q = J - \frac{J - \varepsilon E_b}{1 - \varepsilon} = \frac{(1 - \varepsilon)J}{1 - \varepsilon} - \frac{J - \varepsilon E_b}{1 - \varepsilon} = \frac{-\varepsilon J + \varepsilon E_b}{1 - \varepsilon} = \frac{\varepsilon(E_b - J)}{1 - \varepsilon}$$

Then, for heat flow rate:

$$\dot{Q} = qA = \frac{E_b - J}{\left(\frac{1 - \varepsilon}{\varepsilon A}\right)}$$

Now, let's consider the energy transferred between 2 objects. There are objects 1 and 2, with radiosity J_1 and J_2 ,and area A_1 and A_2.

To find the radiation leaving surface 1 and incident on surface 2, we can use the shape factor. The radiation leaving surface 1 is simply the product of radiosity and surface area. Then, we multiply by the shape factor, which is the portion of radiosity incident on surface 2:

$$radiation\ leaving\ 1\ incident\ on\ 2 = J_1 A_1 F_{1 \to 2}$$

The same logic applies for radiation leaving surface 2 incident on surface 1:

$$radiation\ leaving\ 2\ incident\ on\ 1 = J_2 A_2 F_{2\to 1}$$

To find the net exchange between surface 1 and 2 (or the heat flow rate from 1 to 2), we simply calculate the difference between the two:

$$\dot{Q}_{1\to 2} = J_1 A_1 F_{1\to 2} - J_2 A_2 F_{2\to 1}$$

where $\dot{Q}_{1\to 2}$ is the heat flow rate from surface 1 to surface 2.

We can use the reciprocity relation to simplify this:

$$\dot{Q}_{1\to 2} = J_1 A_1 F_{1\to 2} - J_2 A_1 F_{1\to 2} = A_1 F_{1\to 2}(J_1 - J_2)$$

$$\dot{Q}_{1\to 2} = \frac{J_1 - J_2}{\frac{1}{A_1 F_{1\to 2}}}$$

Electrical circuit analogy

Notice how the equation for heat flow rate out of a surface is very similar to Ohm's law:

$$\dot{Q} = \frac{E_b - J}{\left(\frac{1-\varepsilon}{\varepsilon A}\right)}, \qquad I = \frac{\Delta V}{R}$$

In this case we might consider the heat flow rate as being analogous to current and the difference between black body radiation and the radiation leaving a gray body being analogous to electrical potential difference. In this case, $\frac{1-\varepsilon}{\varepsilon A}$ is analogous to resistance.

For that matter, the equation for heat flow rate is also similar to Ohm's law:

$$\dot{Q}_{1\to 2} = \frac{J_1 - J_2}{\frac{1}{A_1 F_{1\to 2}}}, \qquad I = \frac{\Delta V}{R}$$

In this case, the difference between radiation leaving surfaces 1 and 2 is analogous to potential difference, and resistance is $\frac{1}{A_1 F_{1\to 2}}$.

We can put these resistors in series to represent to full process of energy being transferred from object 1 to 2:

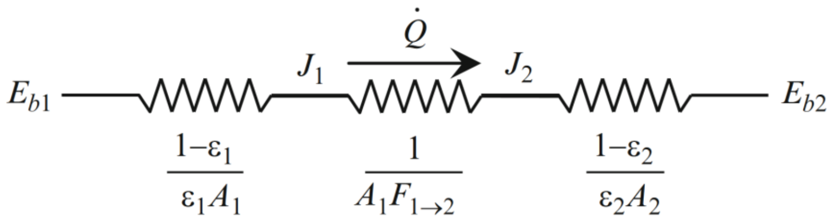


Fig. 34. Equivalent electrical circuit for heat exchange between two surfaces through radiation [1].

In fact, we can create a more complicated circuits to consider systems with more objects. For 3 objects, for example:

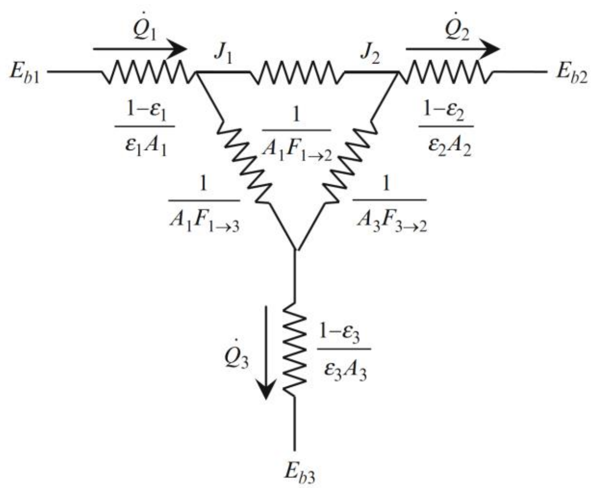


Fig. 35. Equivalent electrical circuit for heat exchange between three surfaces through radiation [1].

Boundary Conditions

To solve the differential equations involved in heat transfer problems, we need some boundary conditions. There are 4 common boundary conditions. We will assume that this "boundary" is at $x = 0$ for the equations.

- **Constant surface temperature**

$$T(x = 0, t) = T_s$$

  Where $T(x = 0, t)$ is the temperature at position $x = 0$ (at the boundary) and at any time t, and T_s is some constant surface temperature.
- **Convection at the surface, fluid temperature far from the surface (T_∞) is constant.**

$$q = -k\frac{\delta T(x = 0,)}{\delta x} = h\left(T_\infty - T(x = 0, t)\right)$$

Where q is the heat flux, k is thermal conductivity, h is heat transfer coefficient. T_∞ is the temperature of the fluid far away from the surface.

- **Constant energy flux at the surface**

$$q = q_x = -k\frac{\delta T(x = 0, t)}{\delta x}$$

Where q_x is some constant heat flux.

- **No heat flux at the surface (insulated surface)**

$$q = 0 = -k\frac{\delta T(x = 0, t)}{\delta x}$$

Macroscopic Approach

Once again, we can approach problems using a "macroscopic" approach, where we ignore spatial variations in the system. This approach is well suited when we only care about energy entering and leaving the system, but not well suited for temperature gradients, whether we are at steady state or not.

Conservation of Energy

Since we are dealing with heat, which is energy, we only need to deal with conservation of energy. As before, we can state conservation of energy as:

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ energy\ \ in\ the\ system \end{Bmatrix} = \begin{Bmatrix} rate\ energy \\ enters\ system \end{Bmatrix} - \begin{Bmatrix} rate\ energy \\ exits\ system \end{Bmatrix} + \begin{Bmatrix} rate\ of\ production \\ of\ energy \end{Bmatrix}$$

We've already done some of the work in the fluid dynamics section, so we can start directly with this formula for conservation of energy

$$\frac{dE}{dt} = \sum_{i}^{num\ inlets/outlets} \pm \int_{A_i} \left(\widehat{U}_i + \widehat{K}_i + \widehat{\Phi}_i + \frac{P_i}{\rho}\right) \rho_i v_i dA + \dot{Q}_s + \dot{Q}_{gen} - \dot{W}_s - \dot{W}_f$$

where E is the energy in the system, t is time, A_i is the cross-sectional area of the inlet or outlet, $\widehat{U}$ is specific internal energy, $\widehat{K}$ is specific kinetic energy, $\widehat{\Phi}$ is specific potential energy, P is pressure at the inlet, ρ is density, $\dot{Q}_s$ is heat flow rate at the surface from conduction/convection/radiation, $\dot{Q}_{gen}$ is rate of heat generation (for example by chemical reactions or metabolism), $\dot{W}_s$ is rate of moving boundary work, and $\dot{W}_f$ is frictional forces at the inlets/outlets. To make notation simpler, we use $\pm$ rather than separating the sum for inlets and outlets, so the $\pm$ is positive if it is an inlet and negative if it is an outlet.

For heat transfer, a lot of these terms aren't important. **Kinetic energy, potential energy, frictional work, and moving boundary work are all negligible. In addition, pressure is of negligible contribution**. This greatly simplifies the equation:

$$\frac{dE}{dt} = \sum_{i}^{num\ inlets/outlets} \pm \int_{A_i} \hat{U}_i \rho_i v_i dA + \dot{Q}_s + \dot{Q}_{gen}$$

We can further assume that **changes in the system energy are only changes in the system's internal energy** (potential and kinetic energy of the system remain constant). Remember that the specific internal energy is:

$$\hat{U} = c_p(T - T_R)$$

where c_p is the specific heat capacity, T is temperature, and T_R is a reference temperature.

Then, we can rewrite the equation, replacing our internal energy terms with their definition and using the mixing cup temperature to get rid of the integrals:

$$\frac{d}{dt}\left(mc_p\bar{T}\right) = \sum_{i}^{num\ inlets/outlets} \pm c_{p,i} T_{m,i} \rho_i v_i A_i + \dot{Q}_s + \dot{Q}_{gen}$$

where $\bar{T}$ is average temperature and T_m is mixing cup temperature.

We can also write this in terms of the mass flow rate, which is $\rho v A$:

$$\frac{d}{dt}\left(mc_p\bar{T}\right) = \sum_{i}^{num\ inlets/outlets} \pm w_i c_{p,i} T_{m,i} + \dot{Q}_s + \dot{Q}_{gen}$$

where w is mass flow rate.

Phase Change

When a heat transfer problem involves phase changes, there are additional considerations. When a material changes phase, it absorbs or releases heat without changing temperature; any energy gained or lost will go towards changing its state. Thus, we must consider the latent heat of the material, which is the energy required for it to change state. These problems will most likely not come up in this course, since they are rather complicated. Still, they are very important, especially in biological contexts as they govern things such as evaporative cooling through sweat.

Lumped Parameter Analysis

In lumped parameter analysis, we have an unsteady state problem where we take a solid object being heated or cooled through convection at its surface, with a constant fluid temperature. Importantly, we assume our object is a lump with uniform temperature. For this to hold, we need the heat transfer inside the object to be much faster than heat transfer at the surface, such that we can assume heat transfer from the core of the object to its surface is instant. This generally holds as long as the Biot number is lower than 0.1.

The Biot number compares the rate of heat conduction within the solid to the rate of heat convection at the surface. Be careful not to confuse it with the Nusselt number, which deals with heat conduction in the **fluid**, not solid. The equations for these numbers are almost identical, but the Nusselt number uses the thermal conductivity of the fluid, while the Biot number uses that of the solid.

$$Bi = \frac{hL}{k_s} = \frac{hV}{k_s A_Q}$$

where Bi is the Biot number, h is the heat transfer coefficient, L is the characteristic length defined as the ratio of volume to area through which heat is transferred $\left(L = \frac{V}{A_Q}\right)$, k_s is the thermal conductivity of the solid, V is the solid volume, A_Q is the surface area through which the solid is heated/cooled.

When the Biot number is very high, the rate of convection is faster that the rate of conduction within the solid, so conduction will not be fast enough to dissipate the heat at the surface, which will create a temperature gradient.

If the system is a solid (no flow in or out), does not generate heat, and temperature is uniform throughout the solid, the conservation of energy equation simplifies to:

$$mc_p \frac{dT}{dt} = \dot{Q}_s$$

We can use Newton's law of cooling:

$$mc_p \frac{dT}{dt} = -hS(T - T_\infty)$$

where h is the heat transfer coefficient, S is the surface area, and T_∞ is the constant temperature of the fluid far away from the surface of the solid.

This is a first-order ODE. To solve it more easily, we can make the substitution $\theta = (T - T_\infty)$, and substitute the derivative with $\frac{d\theta}{dt} = \frac{dT}{dt}$.

$$mc_p \frac{d\theta}{dt} = -hS\theta$$

The solution is:

$$\int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = \int_0^t -\frac{hS}{mc_p} dt$$

$$\ln\left(\left|\frac{\theta}{\theta_i}\right|\right) = -\frac{hS}{mc_p} t$$

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{hS}{mc_p} t}$$

where T_i is the initial temperature.

Electrical circuit analogy

Let's cast our minds far back to RC circuits. RC circuits are simple circuits composed of one capacitor and one resistor. The voltage across the capacitor varied as a function of time following the equation:

$$\frac{V}{V_0} = e^{-\frac{t}{RC}}$$

where V is voltage across capacitor, V_0 is initial voltage across capacitor, R is electrical resistance, and C is electrical capacitance.

Both of our equations are similar. We can define a time constant τ, and see that both functions follow the function $e^{-\frac{t}{\tau}}$. This time constant is:

$$\tau = \frac{1}{RC} = \frac{hS}{mc_p}$$

We had already covered a thermal resistance to convection:

$$R_{T,conv} = \frac{1}{hS}$$

We can also define a thermal capacitance, which is the ability of a material to store heat:

$$C_{thermal} = mc_p$$

So, our lumped parameter analysis was just an RC circuit in the end. We were once again fooled into doing electrical engineering.

Thermal Compartmental analysis

In thermal compartmental analysis, we once again have an unsteady state problem, but we take a well-mixed fluid chamber instead to keep the temperature uniform throughout the fluid. In this case, any fluid leaving the system through outlets would have the same temperature as the system. The conservation of energy equation simplifies to:

$$\frac{d}{dt}\left(mc_pT\right) = \sum_{i}^{num\ inlets} w_i c_{p,i} T_{m,i} - c_pT \sum_{i}^{num\ outlets} w_i + \dot{Q}_s + \dot{Q}_{gen}$$

As an example, let's consider an insulated chamber with no heat generation, a single outlet delivering fluid at a mass flow rate of w and temperature T_{in} and a single outlet with the same mass flow rate as the inlet. c_p will be constant, and the chamber is well-mixed. We wish to know the temperature of the fluid. This situation is very similar to those you have seen in MATH 263, but they may have used concentration of a solute instead of heat.

The equation simplifies to:

$$mc_p \frac{d}{dt}(T) = wc_pT_{in} - wc_pT$$

$$m\frac{dT}{dt} = w(T_{in} - T)$$

Let's use a substitution again, with $z = T_{in} - T$ and $\frac{dz}{dt} = -\frac{dT}{dt}$:

$$-m\frac{dz}{dt} = wz$$

Solving the ODE yields:

$$\frac{T - T_{in}}{T_i - T_{in}} = e^{-\frac{w}{m}t}$$

Multiple System Interactions

You can split a system into multiple interacting subsystems. We have seen how to deal with lumps of uniform temperature and well-mixed chambers. We could combine multiple of them. For example, we could have two connected well-mixed chambers, or a lump within a well-mixed chamber. In this case, we would need to deal with a system of differential equations.

Example:

An object at temperature T_{s0} is dropped in an insulated well-mixed fluid chamber at initial temperature T_{f0}. The Biot number is small, and there are no outlets or inlets. We are interested in the temperature of the solid and fluid over time.

The difference between this situation and the lumped analysis situation is that the fluid temperature can change. This small difference leads to a large increase in our headaches.

Here, the only heat transfer occurring is convection between fluid and solid. Then, by conservation of energy, we have this equation for the solid:

$$m_s c_{ps} \frac{dT_s}{dt} = hS\left(T_f - T_s\right)$$

and this equation for the fluid:

$$m_f c_{pf} \frac{dT_f}{dt} = hS\left(T_s - T_f\right)$$

We need to solve the system of equations:

$$\begin{cases} m_s c_{ps} \dfrac{dT_s}{dt} = hS\left(T_f - T_s\right) \\ m_f c_{pf} \dfrac{dT_f}{dt} = hS\left(T_s - T_f\right) \end{cases}$$

Let's start by isolating T_f in the solid equation:

$$T_f = T_s + \frac{m_s c_{ps}}{hS} \frac{dT_s}{dt}$$

We can substitute it into our fluid equation:

$$m_f c_{pf} \frac{d}{dt}\left(T_s + \frac{m_s c_{ps}}{hS} \frac{dT_s}{dt}\right) = hS\left(T_s - T_s - \frac{m_s c_{ps}}{hS} \frac{dT_s}{dt}\right)$$

$$m_f c_{pf} \frac{T_s}{dt} + \frac{m_f c_{pf} m_s c_{ps}}{hS} \frac{d^2 T_s}{dt^2} = -m_s c_{ps} \frac{dT_s}{dt}$$

$$\frac{m_f c_{pf} m_s c_{ps}}{hS}\frac{d^2T_s}{dt^2} + \left(m_f c_{pf} + m_s c_{ps}\right)\frac{dT_s}{dt} = 0$$

$$\frac{d^2T_s}{dt^2} + \frac{hS}{m_s c_{ps}}\left(1 + \frac{m_s c_{ps}}{m_f c_{pf}}\right)\frac{dT_s}{dt} = 0$$

This is a first order ODE. It may look second order but we can substitute $a = \frac{dT_s}{dt}$:

$$\frac{da}{dt} + \frac{hS}{m_s c_{ps}}\left(1 + \frac{m_s c_{ps}}{m_f c_{pf}}\right)a = 0$$

We can solve this:

$$a = \frac{dT_s}{dt} = C_a e^{-\frac{hS}{m_s c_{ps}}\left(1+\frac{m_s c_{ps}}{m_f c_{pf}}\right)t}$$

We can integrate to find T_s:

$$T_s = C_1 e^{-\frac{hS}{m_s c_{ps}}\left(1+\frac{m_s c_{ps}}{m_f c_{pf}}\right)t} + C_2$$

Note that we simply replaced C_a with C_1 to make the equation less ugly. These constants are related by:

$$C_1 = \frac{C_a}{-\frac{hS}{m_s c_{ps}}\left(1 + \frac{m_s c_{ps}}{m_f c_{pf}}\right)}$$

Now all we have to do is find C_1 and C_2 with our boundary conditions. Which are:

$$T_s(t = 0) = T_{s0}$$

$$T_f(t = 0) = T_{f0}$$

We can directly apply the first boundary condition to the equation we've obtained to find:

$$T_{s0} = C_1 e^0 + C_2 = C_1 + C_2$$

The second boundary condition is trickier. We can first take the solid equation and isolate $\frac{dT_s}{dt}$:

$$\frac{dT_s}{dt} = \frac{hS}{m_s c_{ps}}\left(T_f - T_s\right)$$

Then, at time 0:

$$\frac{dT_s}{dt}\Big|_{t=0} = \frac{hS}{m_s c_{ps}}\left(T_{f0} - T_{s0}\right)$$

We've already found the equation for $\frac{dT_s}{dt}$, so we can find $\frac{dT_s}{dt}\Big|_{t=0}$:

$$\frac{dT_s}{dt}\Big|_{t=0} = C_a e^{-\frac{hS}{m_s c_{ps}}\left(1+\frac{m_s c_{ps}}{m_f c_{pf}}\right)(0)} = C_a = -\frac{hS}{m_s c_{ps}}\left(1 + \frac{m_s c_{ps}}{m_f c_{pf}}\right) C_1$$

Putting the two equations together:

$$-\frac{hS}{m_s c_{ps}}\left(1 + \frac{m_s c_{ps}}{m_f c_{pf}}\right) C_1 = \frac{hS}{m_s c_{ps}}\left(T_{f0} - T_{s0}\right)$$

Then:

$$C_1 = \left(\frac{m_f c_{pf}}{m_f c_{pf} + m_s c_{ps}}\right)\left(T_{f0} - T_{s0}\right), \qquad C_2 = T_{s0} - \left(\frac{m_f c_{pf}}{m_f c_{pf} + m_s c_{ps}}\right)\left(T_{f0} - T_{s0}\right)$$

We can substitute these into the equation to yield:

$$\frac{T_s - T_{s0}}{T_{f0} - T_{s0}} = \left(\frac{m_f c_{pf}}{m_f c_{pf} + m_s c_{ps}}\right)\left(1 - e^{-\frac{hS}{m_s c_{ps}}\left(1+\frac{m_s c_{ps}}{m_f c_{pf}}\right)t}\right)$$

Note that if the mass of the fluid goes to infinity:

$$\frac{T_s - T_{s0}}{T_{f0} - T_{s0}} = 1 - e^{-\frac{hS}{m_s c_{ps}}t}$$

Let's rearrange this a little:

$$\frac{T_s - T_{s0}}{T_{f0} - T_{s0}} - 1 = -e^{-\frac{hS}{m_s c_{ps}}t}$$

$$\frac{T_s - T_{s0}}{T_{f0} - T_{s0}} - \frac{T_{f0} - T_{s0}}{T_{f0} - T_{s0}} = \frac{T_s - T_{f0}}{T_{f0} - T_{s0}} = -e^{-\frac{hS}{m_s c_{ps}}t}$$

$$\frac{T_s - T_{f0}}{T_{s0} - T_{f0}} = e^{-\frac{hS}{m_s c_{ps}}t}$$

which is what we got for lumped parameter analysis.

Shell Balance

When there are spatial variations within the system, such as when the system isn't well-mixed or the Biot number isn't small, the macroscopic approach isn't sufficient. If the problem is one-dimensional and at steady state, then we may use a shell-balance approach.

General Method

1. Define a shell.
    - The shell is a small region of the fluid of interest.
    - When working in cartesian coordinates, place a plane perpendicular to the direction of the temperature gradient, covering the entire cross-sectional area of the object, and another similar plane a little further along the direction of the temperature gradient. Your shell is the region between the planes.
    - In cylindrical and spherical coordinates, the shell will be the space between a cylinder (or sphere), and a slightly larger cylinder (or sphere), making sure that the direction of the temperature gradient is along the radius.
2. Perform an energy balance.
    - Using conservation of energy, list all energy entering or leaving the shell.
    - Divide by the volume of the shell and take the limit as the shell volume goes to zero.
3. Replace the conductive/convective/radiative heat flux equations with Fourier's law, Newton's law, or the Stefan-Boltzmann law.
    - You may need to apply boundary conditions for convection here (i.e. constant surface temperature, constant fluid temperature).
4. Solve the differential equation.
    - Apply Boundary conditions.

Example: Solid Sphere with Heat Generation

Let's take a sphere of radius R with constant, uniform heat generation per unit volume $\dot{q}_{met}$. We are at steady state, and heat only travels in the r-direction.

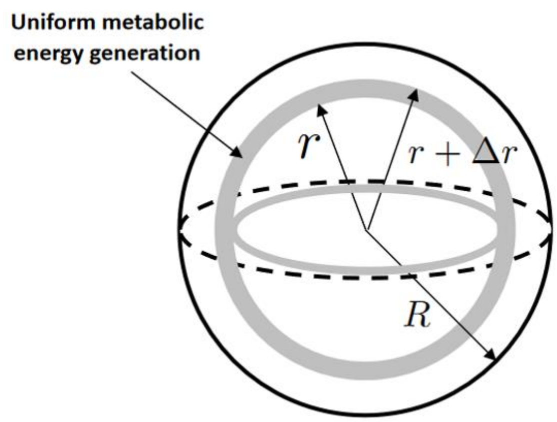


Fig. 36. Diagram of a spherical shell [4].

Here, we have two boundary conditions:

- Constant surface temperature T_r :

$$T(r = R) = T_R$$

- All heat generated in the sphere must leave through the surface of the sphere. Since we are at steady state, there cannot be heat accumulation over time, otherwise the energy of the sphere would change over time. The only way the energy generated in the sphere can leave is through the surface. Thus:

$$q_r(r = R)S = \dot{q}_{met}V_s$$

$$q_r(r = R)(4\pi R^2) = \dot{q}_{met}\left(\frac{4}{3}\pi R^3\right)$$

$$q_r(r = R) = \frac{\dot{q}_{met}R}{3}$$

where q_r is the heat flux through conduction, s is the surface area of the sphere, and V_s is the volume of the sphere.

We set up the shell in spherical coordinates to be the space between two spheres. From conservation of energy:

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ energy\ \ in\ the\ system \end{Bmatrix} = \begin{Bmatrix} rate\ energy \\ enters\ system \end{Bmatrix} - \begin{Bmatrix} rate\ energy \\ exits\ system \end{Bmatrix} + \begin{Bmatrix} rate\ of\ production \\ of\ energy \end{Bmatrix}$$

Since we are within a solid, the only way energy can enter or leave the shell is through conduction. For now, let's leave the conductive heat flux as a variable q_r. We can then obtain the heat flow rate going in or out of the system by multiplying heat flux by area:

$$\begin{Bmatrix} rate\ energy \\ enters\ system \end{Bmatrix} - \begin{Bmatrix} rate\ energy \\ exits\ system \end{Bmatrix} = q_r|_r(4\pi r^2) - q_r|_{r+\Delta r}(4\pi(r + \Delta r)^2)$$

Note that $(r^2 q_r)|_{r+\Delta r} = q_r|_{r+\Delta r}(r+\Delta r)^2$, since the value of r evaluated at position $r + \Delta r$ is $r + \Delta r$. Then, we can simplify the equation to:

$$\begin{Bmatrix} rate\ enegy \\ enters\ system \end{Bmatrix} - \begin{Bmatrix} rate\ energy \\ exits\ system \end{Bmatrix} = 4\pi(r^2 q_r)|_r - 4\pi(r^2 q_r)|_{r+\Delta r}$$

For the energy production in the system, it is simply the heat generation per unit volume $\dot{q}_{met}$ multiplied by the shell volume. The shell volume can be simplified as $4\pi r^2 \Delta r$, using the same logic we used in cylindrical coordinates for shell balance in the fluid dynamics case.

$$\begin{Bmatrix} rate\ of\ production \\ of\ energy \end{Bmatrix} = 4\pi r^2 \Delta r \dot{q}_{met}$$

Putting it all together:

$$0 = 4\pi\left(r^2 q_r\right)|_r - 4\pi\left(r^2 q_r\right)|_{r+\Delta r} + 4\pi r^2 \Delta r \dot{q}_{met}$$

Dividing by $4\pi r^2 \Delta r$:

$$0 = \frac{1}{r^2} \frac{\left(r^2 q_r\right)|_r - \left(r^2 q_r\right)|_{r+\Delta r}}{\Delta r} + \dot{q}_{met}$$

Taking the limit as $\Delta r \to 0$:

$$0 = \frac{1}{r^2} \lim_{\Delta r \to 0} \left(\frac{\left(r^2 q_r\right)|_r - \left(r^2 q_r\right)|_{r+\Delta r}}{\Delta r} \right) + \dot{q}_{met}$$

$$0 = \frac{1}{r^2} \left(-\frac{d}{dr}\left(r^2 q_r\right) \right) + \dot{q}_{met}$$

$$\frac{d}{dr}\left(r^2 q_r\right) = r^2 \dot{q}_{met}$$

Integrating:

$$r^2 q_r = \frac{r^3 \dot{q}_{met}}{3} + C_1$$

$$q_r = \frac{r \dot{q}_{met}}{3} + \frac{C_1}{r^2}$$

We can already apply one of the boundary conditions to get rid of C_1. All heat generated within the sphere must exit through the surface. If we calculated the heat flux at the surface:

$$q_r(r = R) = \frac{R\dot{q}_{met}}{3} + \frac{C_1}{R^2} = \frac{\dot{q}_{met}R}{3}$$

Then, we know that:

$$C_1 = 0$$

Next, we apply Fourier's law to replace q_r:

$$q_r = -k\frac{dT}{dr} = \frac{r\dot{q}_{met}}{3}$$

We can integrate this:

$$T = \frac{r^2\dot{q}_{met}}{6k} + C_2$$

If we use the boundary condition $T(r = R) = T_R$, we obtain:

$$T - T_r = \frac{\dot{q}_{met}}{6k}(R^2 - r^2)$$

Example: Heat exchanger

Let's apply shell balance to a heat exchanger. Some hot fluid enters a pipe at mixing cup temperature $T_{m,i}$. This pipe is immersed within a cold fluid which flows countercurrent to the liquid in the pipe, at a constant temperature far away from the wall T_∞. The heat transfer coefficients are h_{in} within the pipe and h_{out} outside of the pipe. We wish to know the mixing cup temperature T_m of the fluid within the pipe as a function of the distance along the pipe. We are at steady state, the fluid is incompressible, and all properties are independent of temperature. In addition, the pipe wall L is very thin.

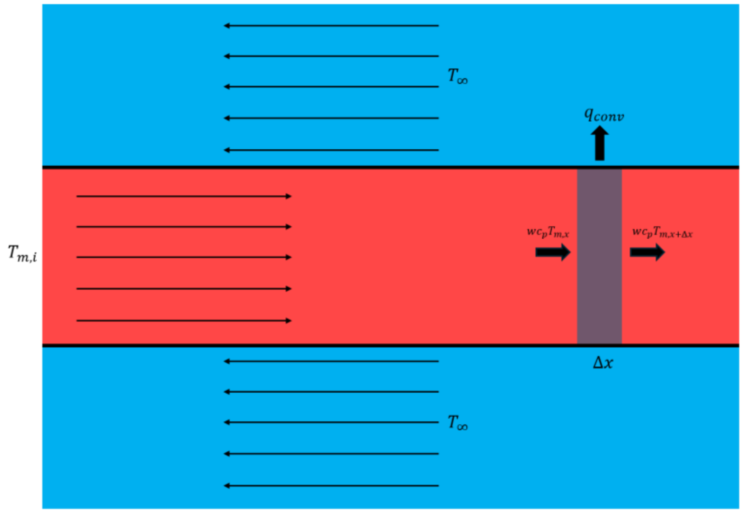


Fig. 37. Diagram of a shell in a heat exchanger problem.

Our boundary conditions are:

- The temperature at the entrance is $T_{m,i}$:
$$T_m(x = 0) = T_{m,i}$$
- The temperature of the fluid far away from the wall is constant T_∞.

Applying the shell balance, we have no heat generation. There are two sources of heat entering/leaving the shell: bulk fluid motion and convection at the wall.

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ energy\ \ in\ the\ system \end{Bmatrix} = \begin{Bmatrix} energy\ loss\ or\ gain \\ from\ bulk\ fluid \end{Bmatrix} + \begin{Bmatrix} energy\ loss\ or\ gain \\ from\ convection \end{Bmatrix}$$

On the right and left wall, there is the bulk flow of fluid at a certain temperature entering or leaving the shell. This contributes the internal energy of the fluid entering or leaving the shell to the overall energy balance:

$$\begin{Bmatrix} energy\ loss\ or\ gain \\ from\ bulk\ fluid \end{Bmatrix} = (wc_pT_m)|_x - (wc_pT_m)|_{x+\Delta x}$$

We can use mass balance to prove that, if the fluid is incompressible, the mass flow rate is constant over x. Then, this simplifies to:

$$\begin{Bmatrix} energy\ loss\ or\ gain \\ from\ bulk\ fluid \end{Bmatrix} = wc_p(T_m|_x - T_m|_{x+\Delta x})$$

In addition, heat is being transferred through the wall of the pipe. To reach the surrounding fluid, heat within the pipe must first cross to the inner surface of the pipe through convection, then from the inner to the outer surface of the pipe through conduction, and finally from the outer surface to the surrounding fluid through convection. We can model this as a series of 3 resistors:

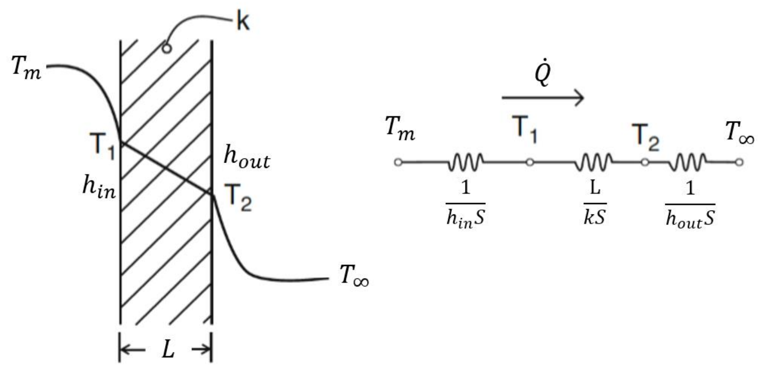


Fig. 38. Diagram of equivalent electrical circuit to heat transport across a heat exchanger [1].

However, since the pipe wall thickness is very thin, we can ignore the resistance due to conduction in the pipe wall. Then, the heat transfer due to convection is:

$$\left\{\begin{matrix} energy\ loss\ or\ gain \\ from\ convection \end{matrix}\right\} = \frac{T_\infty - T_m}{\dfrac{1}{h_{in}S} + \dfrac{1}{h_{out}S}} = \left(\frac{h_{out}h_{in}P\Delta x}{h_{out} + h_{in}}\right)(T_\infty - T_m)$$

where S is the surface area and P is the perimeter.

Putting it all together:

$$0 = wc_p(T_m|_x - T_m|_{x+\Delta x}) + \left(\frac{h_{out}h_{in}P\Delta x}{h_{out} + h_{in}}\right)(T_\infty - T_m)$$

Dividing by Δx and taking the limit as $\Delta x \to 0$:

$$0 = -wc_p\frac{dT_m}{dx} + \left(\frac{h_{out}h_{in}P}{h_{out} + h_{in}}\right)(T_\infty - T_m)$$

$$\frac{dT_m}{dx} = \left(\frac{h_{out}h_{in}}{h_{out} + h_{in}}\right)\left(\frac{P}{wc_p}\right)(T_\infty - T_m)$$

Let's substitute $\theta = T_\infty - T_m$, $\frac{d\theta}{dx} = -\frac{dT_m}{dx}$, and solve using the boundary condition:

$$\frac{d\theta}{dx} = -\left(\frac{h_{out}h_{in}}{h_{out} + h_{in}}\right)\left(\frac{P}{wc_p}\right)\theta$$

$$\int_{\theta_i}^{\theta}\frac{1}{\theta}d\theta = \int_0^x -\left(\frac{h_{out}h_{in}}{h_{out} + h_{in}}\right)\left(\frac{P}{wc_p}\right)dx$$

$$\ln\left(\frac{\theta}{\theta_i}\right) = -\left(\frac{h_{out}h_{in}}{h_{out}+h_{in}}\right)\left(\frac{P}{wc_p}\right)x$$

$$\frac{T_\infty - T_{m,x}}{T_\infty - T_{m,i}} = e^{-\left(\frac{h_{out}h_{in}}{h_{out}+h_{in}}\right)\left(\frac{P}{wc_p}\right)x}$$

General Method

Derivation

Let's start with a shell that is a small rectangular prism with sides of length $\Delta x, \Delta y, \Delta z$ at position (x, y, z), just as before, and apply conservation of energy to it.

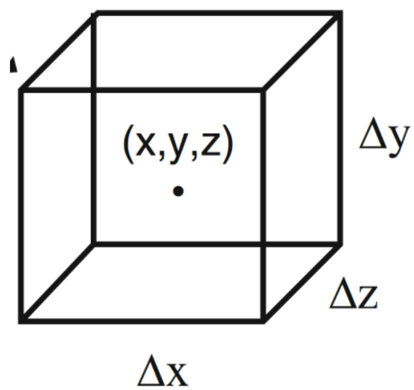


Fig. 39. Shell for general heat transport [1].

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ energy\ \ in\ the\ system \end{Bmatrix} = \begin{Bmatrix} rate\ energy \\ enters\ system \end{Bmatrix} - \begin{Bmatrix} rate\ energy \\ exits\ system \end{Bmatrix} + \begin{Bmatrix} rate\ of\ production \\ of\ energy \end{Bmatrix}$$

We are again ignoring changes in kinetic and potential energy, and assuming all energy change is in internal energy. Thus:

$$\begin{Bmatrix} Rate\ of\ accumulation\ of \\ energy\ \ in\ the\ system \end{Bmatrix} = \frac{\delta}{\delta t}(mU) = \frac{\delta}{\delta t}\left(\rho\Delta x\Delta y\Delta z c_p T\right)$$

where m is mass, U is specific internal energy $\left(U = c_p T\right)$, ρ is density, c_p is specific heat capacity, and T is temperature.

Since we are within an object, the only way energy can enter or leave the shell is through conduction or, if the shell is within a fluid, through bulk fluid carrying energy with it. For each face, we have to consider the effects of conduction and convection. As an example, let's take the bottom and top faces. For now, we keep energy flux due to conduction as a variable q_x, q_y, q_z depending on the direction.

$$\begin{Bmatrix} rate\ energy \\ enters\ bottom \end{Bmatrix} - \begin{Bmatrix} rate\ energy \\ exits\ top \end{Bmatrix} = q_y|_y \Delta x \Delta z - q_y|_{y+\Delta y} \Delta x \Delta z + (\rho v_y U)|_y \Delta x \Delta z - (\rho v_y U)|_{y+\Delta y} \Delta x \Delta z$$

$$\begin{Bmatrix} rate\ energy \\ enters\ bottom \end{Bmatrix} - \begin{Bmatrix} rate\ energy \\ exits\ top \end{Bmatrix} = (q_y + \rho v_y U)|_y \Delta x \Delta z - (q_y + \rho v_y U)|_{y+\Delta y} \Delta x \Delta z$$

$$\begin{Bmatrix} rate\ energy \\ enters\ bottom \end{Bmatrix} - \begin{Bmatrix} rate\ energy \\ exits\ top \end{Bmatrix} = (q_y + \rho v_y c_p T)|_y \Delta x \Delta z - (q_y + \rho v_y c_p T)|_{y+\Delta y} \Delta x \Delta z$$

where v_i is i-direction velocity.

If we do this for the other 4 faces, we are left with:

$$\begin{aligned} \begin{Bmatrix} rate\ enegy \\ enters\ system \end{Bmatrix} - \begin{Bmatrix} rate\ energy \\ exits\ system \end{Bmatrix} \\ &= (q_y + \rho v_y c_p T)|_y \Delta x \Delta z - (q_y + \rho v_y c_p T)|_{y+\Delta y} \Delta x \Delta z + (q_x + \rho v_x c_p T)|_x \Delta y \Delta z \\ &- (q_x + \rho v_x c_p T)|_{x+\Delta x} \Delta y \Delta z + (q_z + \rho v_z c_p T)|_z \Delta x \Delta y \\ &- (q_z + \rho v_z c_p T)|_{z+\Delta z} \Delta x \Delta y \end{aligned}$$

For energy generation, we can use a variable $\dot{q}_{met}$ which is heat generation per unit volume, and multiply by volume:

$$\begin{Bmatrix} rate\ of\ production \\ of\ energy \end{Bmatrix} = \dot{q}_{met} \Delta x \Delta y \Delta z$$

Putting it all together:

$$\begin{aligned} \Delta x \Delta y \Delta z \frac{\delta}{\delta t}(\rho c_p T) \\ &= (q_x + \rho v_x c_p T)|_x \Delta y \Delta z - (q_x + \rho v_x c_p T)|_{x+\Delta x} \Delta y \Delta z + (q_y + \rho v_y c_p T)|_y \Delta x \Delta z \\ &- (q_y + \rho v_y c_p T)|_{y+\Delta y} \Delta x \Delta z + (q_z + \rho v_z c_p T)|_z \Delta x \Delta y \\ &- (q_z + \rho v_z c_p T)|_{z+\Delta z} \Delta x \Delta y + \dot{q}_{met} \Delta x \Delta y \Delta z \end{aligned}$$

Dividing by the volume $\Delta x \Delta y \Delta z$:

$$\begin{aligned} \frac{\delta}{\delta t}(\rho c_p T) &= \frac{(q_x + \rho v_x c_p T)|_x - (q_x + \rho v_x c_p T)|_{x+\Delta x}}{\Delta x} + \frac{(q_y + \rho v_y c_p T)|_y - (q_y + \rho v_y c_p T)|_{y+\Delta y}}{\Delta y} \\ &+ \frac{(q_z + \rho v_z c_p T)|_z - (q_z + \rho v_z c_p T)|_{z+\Delta z}}{\Delta z} + \dot{q}_{met} \end{aligned}$$

Taking the limit as $\Delta x, \Delta y, \Delta z \to 0$:

$$\frac{\delta}{\delta t}(\rho c_p T) = -\frac{\delta}{\delta \mathrm{x}}(q_x + \rho v_x c_p T) - \frac{\delta}{\delta \mathrm{y}}(q_y + \rho v_y c_p T) - \frac{\delta}{\delta \mathrm{z}}(q_z + \rho v_z c_p T) + \dot{q}_{met}$$

$$\frac{\delta}{\delta t}(\rho c_p T) = -\frac{\delta}{\delta \mathrm{x}}(\rho v_x c_p T) - \frac{\delta}{\delta \mathrm{y}}(\rho v_y c_p T) - \frac{\delta}{\delta \mathrm{z}}(\rho v_z c_p T) - \frac{\delta}{\delta \mathrm{x}}(q_x) - \frac{\delta}{\delta \mathrm{y}}(q_y) - \frac{\delta}{\delta \mathrm{z}}(q_z) + \dot{q}_{met}$$

Using Fourier's law $\left(q_i = -k\frac{\delta T}{\delta i}\right)$:

$$\frac{\delta}{\delta t}(\rho c_p T) = -\frac{\delta}{\delta \mathrm{x}}(\rho v_x c_p T) - \frac{\delta}{\delta \mathrm{y}}(\rho v_y c_p T) - \frac{\delta}{\delta \mathrm{z}}(\rho v_z c_p T) + \frac{\delta^2 (kT)}{\delta \mathrm{x}^2} + \frac{\delta^2 (kT)}{\delta \mathrm{y}^2} + \frac{\delta^2 (kT)}{\delta \mathrm{z}^2} + \dot{q}_{met}$$

If c_p, k, ρ are constant:

$$\rho c_p \frac{\delta T}{\delta t} = -\rho c_p \left[v_x \frac{\delta T}{\delta x} + v_y \frac{\delta T}{\delta y} + v_z \frac{\delta T}{\delta z}\right] + k\left[\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} + \frac{\delta^2 T}{\delta z^2}\right] + \dot{q}_{met}$$

Other Coordinate Systems

Cylindrical:

$$\rho c_p \frac{\delta T}{\delta t} = -\rho c_p \left[v_r \frac{\delta T}{\delta r} + \frac{v_\theta}{r}\frac{\delta T}{\delta \theta} + v_z \frac{\delta T}{\delta z}\right] + k\left[\frac{1}{r}\frac{\delta}{\delta r}\left(r\frac{\delta T}{\delta r}\right) + \frac{1}{r^2}\frac{\delta^2 T}{\delta \theta^2} + \frac{\delta^2 T}{\delta z^2}\right] + \dot{q}_{met}$$

Spherical:

$$\rho c_p \frac{\delta T}{\delta t} = -\rho c_p \left[v_r \frac{\delta T}{\delta r} + \frac{v_\theta}{r}\frac{\delta T}{\delta \theta} + \frac{v_\phi}{r\,sin\,\theta}\frac{\delta T}{\delta \phi}\right] + k\left[\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta T}{\delta r}\right) + \frac{1}{r^2 \sin\theta}\frac{\delta}{\delta \theta}\left(\sin\theta\frac{\delta T}{\delta \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\delta^2 T}{\delta \phi^2}\right] + \dot{q}_{met}$$

Additional Topics

Fins

A fin is an extended surface which increases the surface area through which heat transfer happens. Blades on an engine, fingers, and CPU air coolers are all example of fins. At steady state, if we assume temperature only varies along the length of the fin, we can describe fins using a shell balance approach. The shell will be the distance between two planes positioned along the length of the fin.

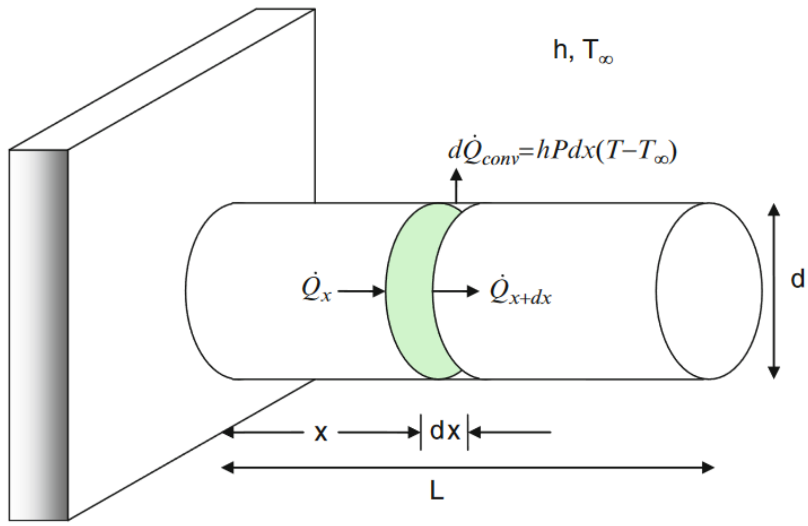


Fig. 40. Shell for a fin and heat inflow and outflow through conduction and convection at the surface [1].

There are two sources of heat transfer and no heat generation. At the left and right sides of the shell, heat enters and leaves through conduction. At the top and bottom of the shell, heat leaves through convection (we assume radiation is much less important than convection). Then, from conservation of energy:

$$0 = \dot{Q}|_x - \dot{Q}|_{x+\Delta x} - \dot{Q}_{conv}$$

where $\dot{Q}$ is the heat flow rate from conduction, and $\dot{Q}_{conv}$ is the heat flow rate from convection.

We can replace the convection term with Newton's law, divide by Δx, and take the limit as $\Delta x \to 0$:

$$0 = \dot{Q}|_x - \dot{Q}|_{x+\Delta x} - hP\Delta x(T - T_\infty)$$

$$0 = \frac{\dot{Q}|_x - \dot{Q}|_{x+\Delta x}}{\Delta x} - hP(T - T_\infty)$$

$$\frac{d\dot{Q}}{dx} = hP(T - T_\infty)$$

where h is the heat transfer coefficient, P is the perimeter of the fin, T is temperature, and T_∞ is the temperature of the fluid far from the fin.

If we replace the conductive heat flow rate with Fourier's law, and assume the cross-sectional area and thermal conductivity are independent of x:

$$\frac{d}{dx}\left(kA_c\frac{dT}{dx}\right) = hP(T - T_\infty)$$

$$\frac{d^2T}{dx^2} = \frac{hP}{kA_c}(T - T_\infty)$$

where A_c is the cross-sectional area and k is the thermal conductivity.

We'll simplify this by noting that $\frac{P}{A_c} = \frac{\pi d}{\frac{\pi d^2}{4}} = \frac{4}{d}$, and by replacing $\frac{hP}{kA_c} = \frac{4h}{kd}$ with a variable m^2:

$$\frac{d^2T}{dx^2} = m^2(T - T_\infty), \qquad m^2 = \frac{4h}{kd}$$

Now all that is left is to solve this. To do this, we need 2 boundary conditions. The first is obvious: the temperature at the base of the fin is the temperature of the bulk material from which the fin extends, which we can call T_0:

$$T(x = 0) = T_0$$

The second boundary concerns the end of the tip, at length L, and depends on the situation. These are three different second boundary conditions, which give different results. Here, the conditions and their results are listed. The problem will be solved for the first listed boundary condition further below, and you should be able to do the second by yourself, but the third is just too long, so only the result will be given.

1. **The tip is insulated**: no heat transfer at the end of the fin.

$$\dot{Q}|_L = k\frac{dT}{dx}|_L = 0$$

The result is:

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = \frac{\cosh\big(m(L - x)\big)}{\cosh(mL)}, \qquad m^2 = \frac{4h}{kd}$$

2. **The fin is very long**: there is enough surface area for convection such that the temperature at the end reaches the fluid temperature at infinity:

$$T(x = \infty) = T_\infty$$

The result is:

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = e^{-mx}, \qquad m^2 = \frac{4h}{kd}$$

3. **The fin is of finite length:** there is convection at the end.

$$-kA_c\frac{dT}{dx}|_L = hA_c(T(x = L) - T_\infty)$$

The result is:

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = \frac{\cosh\big(m(L-x)\big) + \frac{h}{mk}\sinh\big(m(L-x)\big)}{\cosh(mL) + \frac{h}{mk}\sinh(mL)}, \qquad m^2 = \frac{4h}{kd}$$

Solving for the first case

We wish to solve the first case, where the tip is insulated. Thus, we want to solve:

$$\frac{d^2T}{dx^2} = m^2(T - T_\infty), \qquad m^2 = \frac{4h}{kd}$$

with boundary conditions:

$$T(x=0) = T_0$$

$$\frac{dT}{dx}|_L = 0$$

We can start by changing variables $\theta = T - T_\infty, \frac{d\theta}{dt} = \frac{dT}{dx}$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

This is a second order ODE that we should be able to solve. The characteristic equation is:

$$r^2 - m^2 = 0$$

And the roots are $\pm m$, which we know is positive since $\frac{4h}{kd}$ must be positive. So the solution is:

$$\theta(x) = C_1 e^{-mx} + C_2 e^{mx}$$

Now let's apply those boundary conditions. For the first BC:

$$\theta(x=0) = T_0 - T_\infty = C_1 - C_2$$

$$C_2 = (T_0 - T_\infty) - C_1$$

For the second BC, we first find the derivative of $\theta(x)$:

$$\frac{d\theta}{dx} = -C_1 m e^{-mx} + mC_2 e^{mx}$$

At $x = L$:

$$\frac{d\theta}{dx}|_L = \frac{dT}{dx}|_L = 0 = -C_1 m e^{-mL} + mC_2 e^{mL}$$

$$0 = -C_1 e^{-mL} + C_2 e^{mL}$$

Replacing C_2 with the expression we found with the first BC:

$$0 = -C_1 e^{-mL} + \big((T_0 - T_\infty) - C_1\big) e^{mL}$$

$$C_1 e^{-mL} + C_1 e^{mL} = (T_0 - T_\infty) e^{mL}$$

$$C_1 = \frac{(T_0 - T_\infty) e^{mL}}{e^{-mL} + e^{mL}}$$

We can simplify this a little with the definition $\cosh x = \frac{e^x + e^{-x}}{2}$:

$$C_1 = \frac{(T_0 - T_\infty) e^{mL}}{2\cosh(mL)}$$

Then, we can find C_2:

$$C_2 = (T_0 - T_\infty) - \frac{(T_0 - T_\infty) e^{mL}}{2\cosh(mL)}$$

Putting this back in the equation:

$$\theta(x) = \frac{(T_0 - T_\infty) e^{mL}}{2\cosh(mL)} e^{-mx} + \left((T_0 - T_\infty) - \frac{(T_0 - T_\infty) e^{mL}}{2\cosh(mL)}\right) e^{mx}$$

Let's rearrange this:

$$\theta(x) = \frac{(T_0 - T_\infty) e^{mL}}{2\cosh(mL)} e^{-mx} - \frac{(T_0 - T_\infty) e^{mL}}{2\cosh(mL)} e^{mx} + (T_0 - T_\infty) e^{mx}$$

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = \frac{e^{mL} e^{-mx} - e^{mL} e^{mx}}{2\cosh(mL)} + e^{mx}$$

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = \frac{e^{mL} e^{-mx} - e^{mL} e^{mx}}{2\cosh(mL)} + \frac{e^{mx}(2\cosh mL)}{2\cosh(mL)}$$

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = \frac{e^{mL} e^{-mx} - e^{mL} e^{mx}}{2\cosh(mL)} + \frac{e^{mx} e^{mL} + e^{mx} e^{-mL}}{2\cosh(mL)}$$

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = \frac{e^{mL} e^{-mx} + e^{mx} e^{-mL}}{2\cosh(mL)} = \frac{e^{m(L-x)} + e^{-m(L-x)}}{2\cosh(mL)}$$

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = \frac{\cosh\big(m(L - x)\big)}{\cosh(mL)}$$

Unsteady State Symmetrical Internal Thermal Gradients and Heisler Charts

Let's consider a symmetrical slab of thickness $2L$, with no internal heat generation, and try to obtain the temperature distribution along its thickness. We are not at steady state. We will treat this slab as being much taller and wider than its thickness, such that we can consider it a 1D problem.

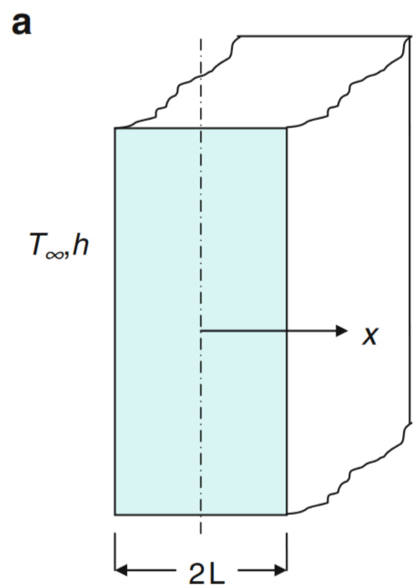


Fig. 41. Diagram of a symmetrical slab with convection at the surface [1].

Let's use the general equation.

$$\rho c_p \frac{\delta T}{\delta t} = -\rho c_p \left[v_x \frac{\delta T}{\delta x} + v_y \frac{\delta T}{\delta y} + v_z \frac{\delta T}{\delta z}\right] + k \left[\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} + \frac{\delta^2 T}{\delta z^2}\right] + \dot{q}_{met}$$

We can remove all convective terms $\left(\rho c_p \left[v_x \frac{\delta T}{\delta x} + v_y \frac{\delta T}{\delta y} + v_z \frac{\delta T}{\delta z}\right]\right)$ since we are in a solid, all terms with y or z since we are dealing with a 1D problem, and the heat generation term $\dot{q}_{met}$.

$$\rho c_p \frac{\delta T}{\delta t} = k \frac{\delta^2 T}{\delta x^2}$$

$$\frac{\delta T}{\delta t} = \alpha \frac{\delta^2 T}{\delta x^2}, \qquad \alpha = \frac{k}{\rho c_p}$$

where α is thermal diffusivity.

We need three boundary conditions. For the first, we can say that the initial temperature of the slab was a uniform T_i. For the second, the fact it is symmetrical means that the maximum or minimum temperature must be at its center. For the final boundary condition, we can say that the slab is exchanging heat with the environment through convection at the edges, and the temperature of the environment is T_∞.

$$T(x, t = 0) = T_i$$

$$\frac{\delta T}{\delta x}|_{x=0} = 0$$

$$-\frac{\delta T}{\delta x}|_{x=L} = h(T(x = L, t) - T_\infty)$$

Now, let's scale the equation. Unlike previously, the goal of scaling won't be to find which terms can be ignored. Instead, we want to write the equation in dimensionless terms both to make it easier to solve and to make this equation applicable to a wide range of situations. As we will see later, the equation has already been solved by someone else who made handy charts; but they have solved it in terms of dimensionless numbers to ensure that their charts could be used for any symmetrical slab with no heat generation and these boundary conditions.

We scale temperature with the ratio of the difference between internal and external temperature, and the difference between initial temperature and external temperature.

$$\theta^* = \frac{T - T_\infty}{T_i - T_\infty}$$

For the length, we can simply divide it by the length of the slab.

$$x^* = \frac{x}{L}$$

For the time, we can divide it by the Fourier number, which is the ratio of time to thermal diffusion time $Fo = \frac{\alpha t}{L^2}$. If the Fourier number is around 1, heat should have had time to reach length L.

$$t^* = Fo = \frac{\alpha t}{L^2}$$

That takes care of all our variables. We can replace the variables in our differential equation:

$$\frac{(T_i - T_\infty)}{\left(\frac{L^2}{\alpha}\right)} \frac{\delta \theta^*}{\delta t^*} = \alpha \frac{(T_i - T_\infty)}{L^2} \frac{\delta^2 \theta^*}{\delta x^{*2}}$$

$$\frac{\delta \theta^*}{\delta t^*} = \frac{\delta^2 \theta^*}{\delta x^{*2}}$$

As we can see, this gets rid of the α term.

For the boundary conditions:

$$\theta^*(x^*.0) = 1$$

$$\frac{\delta\theta^*}{\delta x^*}|_{x^*=0} = 0$$

$$-\frac{\delta\theta^*}{\delta x^*}|_{x^*=1} = Bi\theta^*(1, Fo), \qquad Bi = \frac{hL}{k_s}$$

The equation we have to solve is the heat equation, which you have probably seen in MATH 264. The answer is an infinite Fourier series of the form:

$$\theta^* = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 Fo} \cos(\lambda_n x^*), \qquad C_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin 2\lambda_n}, \qquad \lambda_n \tan\lambda_n = Bi$$

As you can see, this is a very long equation. Worse, it needs infinite terms. However, we are engineers, and we don't need exact solutions. Our kind and generous friend Heisler approximated these and made 2 simple charts. The first shows the temperature at the center as a function of time, and the second shows the temperature inside the slab relative to the center temperature. Heisler only used one term, since the poor guy was doing this back in 1947, so the actual Heisler charts are only good enough if the Fourier number is higher than 0.2 ($Fo > 0.2$). In the age of supercomputers in our pockets, more precise "Heisler charts" are available, including in the Roselli and Diller textbook.

Unsteady State Semi-infinite Internal Thermal Gradients

We are again preoccupied with the unsteady state temperature distribution within an object, but instead of treating it as symmetrical, we will assume it is semi-infinite, so the object starts at $x = 0$ and goes on to infinitely high x. Our starting equation is the same as before:

$$\frac{\delta T}{\delta t} = \alpha\frac{\delta^2 T}{\delta x^2}, \qquad \alpha = \frac{k}{\rho c_p}$$

Our boundary conditions are different. The first is, once again, that the material is at a uniform initial temperature T_i. The second stems from the semi-infinite geometry: since the object is infinitely long, we can assume that heat will never have enough time to reach the infinitely far end of the object, so it will always stay at the same temperature T_i.

$$T(x, t = 0) = T_i$$

$$T(x=\infty,t)=T_i$$

We need a third boundary condition. As in the fin case, we have three possibilities for this third boundary conditions. The boundary conditions and their respective results will be listed below. All of these solutions use the error function, defined as:

$$\operatorname{erf} x=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2}\,d\eta$$

Table 8: Solutions to the unsteady state semi-infinite thermal gradient problem for multiple boundary conditions

Boundary condition	Result
Constant surface temperature T_s $T(x=0,t)=T_s$	$\frac{T(x,t)-T_s}{T_i-T_s}=\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$
Constant heat flux at the surface q_s $-k\frac{\delta T}{\delta x}\vert_{x=0}=q_s$	$T(x,t)-T_i=\frac{2q_s\sqrt{\frac{\alpha t}{\pi}}}{k}e^{-\frac{x^2}{4\alpha t}}-\frac{q_s x}{k}\left(1-\operatorname{erf}\frac{x}{2\sqrt{\alpha t}}\right)$
Convection at the surface $-k\frac{\delta T}{\delta x}\vert_{x=0}=h(T(x=0,t)-T_\infty)$	$\frac{T(x,t)-T_i}{T_\infty-T_i}=1-\operatorname{erf}\frac{x}{2\sqrt{\alpha t}}$

Appendix

Table of Dimensionless Numbers

Dimensionless Number	Equation	Ratio
Biot Number	$Bi = \frac{hL}{k_s}$	Ratio of thermal convection at the surface to conduction inside an object
Fourier number	$Fo = \frac{\alpha t}{L^2}$	Ratio of time elapsed to time scale for heat diffusion
Grashof number	$Gr_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2}$	Ratio of buoyant to viscous forces
Graetz number	$Gz_d = \frac{d}{L} Re_d Pr$	Ratio of heat advection to conduction for entrance length
Nusselt number	$Nu = \frac{hL}{k_f}$	Ratio of thermal convection at the surface to conduction inside the fluid
Prandtl number	$Pr = \frac{c_p \mu}{k_f} = \frac{\nu}{\alpha}$	Ratio of momentum diffusivity to thermal diffusivity
Rayleigh number	$Ra_L = \frac{g\beta(T_s - T_\infty)L^3}{\alpha\nu}$ $Ra_L = Gr_L Pr$	Ratio of free convection to thermal diffusion
Reynolds number	$Re_L = \frac{\rho v L}{\mu} = \frac{vL}{\nu}$	Ratio of inertial forces to viscous forces

Substantial derivative

Let's imagine a skydiver (you can consider this diver to be Super Grover). We wish to know the change in temperature this diver feels as they fall. The ambient temperature depends both on time, so temperature can change throughout the day, and space, so the temperature can be colder at higher altitudes for example. Then, the temperature the diver feels depends on their location in time and space, and so how the temperature they feel changes depends on how they move through time and space. Let's derive the substantial derivative using the example. The change in temperature the diver feels with respect to time is the substantial derivate $\frac{DT}{Dt}$.

$$\frac{dT}{dt} = \frac{DT}{Dt}$$

We can use the multivariate chain rule to split the total derivative into its partial derivatives:

$$\frac{DT}{Dt} = \frac{dT(t,x,y,z)}{dt} = \frac{dt}{dt}\frac{\delta T(t,x,y,z)}{\delta t} + \frac{dx}{dt}\frac{\delta T(t,x,y,z)}{\delta x} + \frac{dy}{dt}\frac{\delta T(t,x,y,z)}{\delta y} + \frac{dz}{dt}\frac{\delta T(t,x,y,z)}{\delta z}$$

$\frac{dt}{dt}$ is obviously just one, and $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ depend on the path that the diver takes, and are the x, y, z velocity of the diver v_x, v_y, v_z. This simplifies to:

$$\frac{DT}{Dt} = \frac{\delta T(t,x,y,z)}{\delta t} + v_x \frac{\delta T(t,x,y,z)}{\delta x} + v_y \frac{\delta T(t,x,y,z)}{\delta y} + v_z \frac{\delta T(t,x,y,z)}{\delta z}$$

We can simplify this using a dot product:

$$\frac{DT}{Dt} = \frac{\delta T}{\delta t} + \langle v_x, v_y, v_z \rangle \cdot \left\langle \frac{\delta T}{\delta x}, \frac{\delta T}{\delta y}, \frac{\delta T}{\delta z} \right\rangle$$

$$\frac{DT}{Dt} = \frac{\delta T}{\delta t} + \vec{v} \cdot \nabla T$$

We can generalize this for any scalar or vector quantity Φ, not just temperature:

$$\frac{D\Phi}{Dt} = \frac{\delta \Phi}{\delta t} + \vec{v} \cdot \nabla \Phi$$

If we were to apply this to velocity:

$$\frac{D\vec{v}}{Dt} = \frac{\delta \vec{v}}{\delta t} + \vec{v} \cdot \nabla \vec{v}$$

We can express the velocity a particle feels from knowledge of the velocity vector field. We can also express the Navier-Stokes equation more concisely using the substantial derivative.

$$\rho \frac{D\vec{v}}{Dt} = \mu \nabla^2 \vec{v} - \nabla P + \rho \vec{g}$$

Summary of Equations

Fluid Dynamics

Name	Equation
Newton's law of viscosity	$$\tau_{yx} = \mu \frac{\delta v_x}{\delta y}$$
Simplified conservation of mass	$$\frac{dm}{dt} = \sum_{i=1}^{num\ inlets} \rho\langle v_{in}\rangle A_{in} - \sum_{j=1}^{num\ outlets} \rho\langle v_{out}\rangle A_{out}$$ Assumptions: uniform density across the inlets and outlets, impermeable walls
Simplified conservation of momentum	$$\frac{d\vec{p}}{dt} = \sum_{i=1}^{num\ inlets} \left(\rho_i K_{2i}\langle v\rangle_i^{\ 2} + P_i\right)A_i\vec{e_i} - \sum_{j=1}^{num\ outlets} \left(\rho_j K_{2j}\langle v\rangle_j^{\ 2} + P_j\right)A_j\vec{e_j} + m\vec{g} - \vec{R}$$ Assumptions: density and pressure uniform at entrances and exits.
Engineering Bernoulli	$$\frac{\dot{W}_s}{w} + \frac{E_V}{w} = \frac{1}{2}\left(K_{3in}\langle v_{in}\rangle^2 - K_{3out}\langle v_{out}\rangle^2\right) + g(h_{in} - h_{out}) + \frac{(P_{in} - P_{out})}{\rho}$$ Assumptions: steady state, isothermal, incompressible fluid, no chemical reactions or heat through surfaces, energy and pressure are uniform across cross-section, single inlet and outlet.
K constant	$$K_i = \frac{\langle v^i\rangle}{\langle v\rangle^i}$$
Friction loss	$$\frac{E_v}{w} = 4f\frac{L}{D_h}\left(\frac{1}{2}\langle v\rangle^2\right)$$ $$\frac{E_V}{w} = \sum_{i=1}^{num\ fittings} K_{w,i}\frac{1}{2}\langle v_{out,i}\rangle^2 + \sum_{j=1}^{num\ conduits} 4f_j\frac{L_j}{D_{h,j}}\frac{1}{2}\langle v_j\rangle^2$$
Compliance	$$C = \frac{dV}{dP_{tm}}$$
Electrical circuit analogy to fluid flow	$$Q_V = \frac{\Delta P}{R_f}$$ $$R_f = \frac{8\mu L}{\pi R^4}\ (circular\ conduit), \qquad R_f = \frac{12\mu L}{Bd^3 M_0}(noncircular\ conduits)$$
Continuity equation and Navier-Stokes	$$\nabla\cdot\vec{v} = 0$$ $$\rho\frac{D\vec{v}}{Dt} = \mu\nabla^2\vec{v} - \nabla P + \rho\vec{g}$$ Assumptions: isotropic, incompressible, Newtonian fluid.

Navier-Stokes in all coordinate systems

Cartesian

$$\frac{\delta v_x}{\delta x}+\frac{\delta v_y}{\delta y}+\frac{\delta v_z}{\delta z}=0$$

$$\rho\frac{\delta v_x}{\delta t}+\rho\left(v_x\frac{\delta v_x}{\delta x}+v_y\frac{\delta v_x}{\delta y}+v_z\frac{\delta v_x}{\delta z}\right)=\mu\left(\frac{\delta^2 v_x}{\delta x^2}+\frac{\delta^2 v_x}{\delta y^2}+\frac{\delta^2 v_x}{\delta z^2}\right)-\frac{\delta P}{\delta x}+\rho g_x$$

$$\rho\frac{\delta v_y}{\delta t}+\rho\left(v_x\frac{\delta v_y}{\delta x}+v_y\frac{\delta v_y}{\delta y}+v_z\frac{\delta v_y}{\delta z}\right)=\mu\left(\frac{\delta^2 v_y}{\delta x^2}+\frac{\delta^2 v_y}{\delta y^2}+\frac{\delta^2 v_y}{\delta z^2}\right)-\frac{\delta P}{\delta y}+\rho g_y$$

$$\rho\frac{\delta v_z}{\delta t}+\rho\left(v_x\frac{\delta v_z}{\delta x}+v_y\frac{\delta v_z}{\delta y}+v_z\frac{\delta v_z}{\delta z}\right)=\mu\left(\frac{\delta^2 v_z}{\delta x^2}+\frac{\delta^2 v_z}{\delta y^2}+\frac{\delta^2 v_z}{\delta z^2}\right)-\frac{\delta P}{\delta z}+\rho g_z$$

Cylindrical

$$\frac{1}{r}\frac{\delta}{\delta r}(rv_r)+\frac{1}{r}\frac{\delta}{\delta\theta}(v_\theta)+\frac{\delta}{\delta z}(v_z)=0$$

$$\rho\frac{\delta v_r}{\delta t}+\rho\left(v_r\frac{\delta v_r}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_r}{\delta\theta}-\frac{v_\theta^2}{r}+v_z\frac{\delta v_r}{\delta z}\right)=\mu\left(\frac{\delta}{\delta r}\left(\frac{1}{r}\frac{\delta}{\delta r}(rv_r)\right)+\frac{1}{r^2}\frac{\delta^2 v_r}{\delta\theta^2}-\frac{2}{r^2}\frac{\delta v_\theta}{\delta\theta}+\frac{\delta^2 v_r}{\delta z^2}\right)-\frac{\delta P}{\delta r}+\rho g_r$$

$$\rho\frac{\delta v_\theta}{\delta t}+\rho\left(v_r\frac{\delta v_\theta}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_\theta}{\delta\theta}-\frac{v_r v_\theta}{r}+v_z\frac{\delta v_\theta}{\delta z}\right)=\mu\left(\frac{\delta}{\delta r}\left(\frac{1}{r}\frac{\delta}{\delta r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\delta^2 v_\theta}{\delta\theta^2}+\frac{2}{r^2}\frac{\delta v_r}{\delta\theta}+\frac{\delta^2 v_\theta}{\delta z^2}\right)-\frac{1}{r}\frac{\delta P}{\delta\theta}+\rho g_\theta$$

$$\rho\frac{\delta v_z}{\delta t}+\rho\left(v_r\frac{\delta v_z}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_z}{\delta\theta}+v_z\frac{\delta v_z}{\delta z}\right)=\mu\left(\frac{1}{r}\frac{\delta}{\delta r}\left(r\frac{\delta v_z}{\delta r}\right)+\frac{1}{r^2}\frac{\delta^2 v_z}{\delta\theta^2}+\frac{\delta^2 v_z}{\delta z^2}\right)-\frac{\delta P}{\delta z}+\rho g_z$$

Spherical

$$\frac{1}{r^2}\frac{\delta}{\delta r}(r^2 v_r)+\frac{1}{r\sin\theta}\frac{\delta}{\delta\theta}(v_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\delta}{\delta\phi}(v_\phi)=0$$

$$\rho\frac{\delta v_r}{\delta t}+\rho\left(v_r\frac{\delta v_r}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_r}{\delta\theta}-\frac{v_\phi^2+v_\theta^2}{r}+\frac{v_\phi}{r\sin\theta}\frac{\delta v_r}{\delta\phi}\right)=\mu\left(\nabla^2 v_r-\frac{2}{r^2}\left(v_r+\frac{\delta v_\theta}{\delta\theta}+v_\theta\cot\theta+\frac{1}{\sin\theta}\frac{\delta v_\phi}{d\phi}\right)\right)-\frac{\delta P}{\delta r}+\rho g_r$$

$$\rho\frac{\delta v_\theta}{\delta t}+\rho\left(v_r\frac{\delta v_\theta}{\delta r}+\frac{v_\theta}{r}\left(v_r+\frac{\delta v_\theta}{\delta\theta}\right)+\frac{v_\phi}{r\sin\theta}\left(\frac{\delta v_\theta}{\delta\phi}-v_\phi\cos\theta\right)\right)$$
$$=\mu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\delta v_r}{\delta\theta}-\frac{1}{r^2\sin^2\theta}(v_\theta+2\cos\theta\frac{\delta v_\phi}{\delta\phi})\right)-\frac{1}{r}\frac{\delta P}{\delta\theta}+\rho g_\theta$$

$$\rho\frac{\delta v_\phi}{\delta t}+\rho\left(v_r\frac{\delta v_\phi}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_\phi}{\delta\theta}+\frac{v_\phi}{r\sin\theta}\left(\frac{\delta v_\phi}{\delta\phi}+v_r\sin\theta+v_\theta\cos\theta\right)\right)$$
$$=\mu\left(\nabla^2 v_\phi-\frac{1}{r^2\sin^2\theta}\left(v_\phi-2\sin\theta\frac{\delta v_r}{\delta\phi}-2\cos\theta\frac{\delta v_\theta}{\delta\phi}\right)\right)-\frac{1}{r\sin\theta}\frac{\delta P}{\delta z}+\rho g_\phi$$

Heat Transfer

Name	Equation
Fourier's law of conduction	$q = \frac{\dot{Q}}{S} = k\frac{\delta T}{\delta x}$ $q = \frac{\dot{Q}}{S} = k\frac{\delta T}{\delta r}$
Newton's law of cooling	$q = \frac{\dot{Q}}{S} = h(T_s - T_\infty)\ (external\ flow)$ $q = \frac{\dot{Q}}{S} = h(T_s - T_m)\ (internal\ flow)$
Radiative heat transfer	$q = \frac{\dot{Q}}{S} = \varepsilon\sigma T^4$
Electrical circuit analogy	$\dot{Q} = \frac{\Delta T}{R_T}$
Resistances (conduction and convection)	$R_T = \frac{L}{Ak}\ (cartesian\ conduction),\quad R_T = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi kL}\ (cylindrical\ conduction)$ $R_T = \frac{1}{hS}\ (convection)$
Radiation electrical circuit analogy	$\dot{Q} = \frac{E_b - J}{\left(\frac{1-\varepsilon}{\varepsilon A}\right)}$ $\dot{Q}_{1\to 2} = \frac{J_1 - J_2}{\frac{1}{A_1 F_{1\to 2}}}$
Macroscopic conservation of mass	$\frac{d}{dt}(mc_p\bar{T}) = \sum_i^{num\ inlets} w_i c_{p,i} T_{m,i} - \sum_i^{num\ outlets} w_i c_{p,i} T_{m,i} + \dot{Q}_s + \dot{Q}_{gen}$
General cartesian conservation of mass	$\rho c_p \frac{\delta T}{\delta t} = -\rho c_p\left[v_x\frac{\delta T}{\delta x} + v_y\frac{\delta T}{\delta y} + v_z\frac{\delta T}{\delta z}\right] + k\left[\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} + \frac{\delta^2 T}{\delta z^2}\right] + \dot{q}_{met}$

General conservation of energy in all coordinate systems

Cartesian

$$\rho c_p \frac{\delta T}{\delta t} = -\rho c_p \left[v_x \frac{\delta T}{\delta x} + v_y \frac{\delta T}{\delta y} + v_z \frac{\delta T}{\delta z}\right] + k\left[\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} + \frac{\delta^2 T}{\delta z^2}\right] + \dot{q}_{met}$$

Cylindrical

$$\rho c_p \frac{\delta T}{\delta t} = -\rho c_p \left[v_r \frac{\delta T}{\delta r} + \frac{v_\theta}{r}\frac{\delta T}{\delta \theta} + v_z \frac{\delta T}{\delta z}\right] + k\left[\frac{1}{r}\frac{\delta}{\delta r}\left(r\frac{\delta T}{\delta r}\right) + \frac{1}{r^2}\frac{\delta^2 T}{\delta \theta^2} + \frac{\delta^2 T}{\delta z^2}\right] + \dot{q}_{met}$$

Spherical

$$\rho c_p \frac{\delta T}{\delta t} = -\rho c_p \left[v_r \frac{\delta T}{\delta r} + \frac{v_\theta}{r}\frac{\delta T}{\delta \theta} + \frac{v_\phi}{r\, sin\, \theta}\frac{\delta T}{\delta \phi}\right] + k\left[\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta T}{\delta r}\right) + \frac{1}{r^2 \sin\theta}\frac{\delta}{\delta \theta}\left(\sin\theta\frac{\delta T}{\delta \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\delta^2 T}{\delta \phi^2}\right] + \dot{q}_{met}$$

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